

# Shape optimisation with nonsmooth cost functions: from theory to numerics

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## Abstract

This paper is concerned with the study of a class of nonsmooth cost functions subject to a quasi-linear PDE in Lipschitz domains in dimension two. We derive the Eulerian semi-derivative of the cost function by employing the averaged adjoint approach and maximal elliptic regularity. Furthermore we characterise stationary points and show how to compute steepest descent directions theoretically and practically. Finally, we present some numerical results for a simple toy problem and compare them with the smooth case. We also compare the convergence rates and obtain higher rates in the nonsmooth case.

**Keywords:** shape optimization, nonsmooth cost functions, PDE constraints

**AMS classification:** 49Q10, 49Q12, 35J62, 49K20, 49K40, 49J52

## Introduction

The main object of shape optimisation is the minimisation of a cost or shape function with respect to a design variable. In applications the design variable may be the bodywork of a car or aircraft, but also the shape of antennas or inductor coils are possible design variables. The shape function may be the compliance, drag, friction or any other physically relevant quantities. Mathematically speaking the *design variable* is a subset of the Euclidean space admitting a certain regularity reflecting the smoothness of the design and a *shape/cost function* is a real-valued mapping on the design variables.

While there exists a huge body of research on the topic of smooth shape optimisation problems, see [7, 24, 18, 15] and references therein, the work on nonsmooth problems is far less complete. By a smooth shape optimisation problem we understand that the cost function and the constraints (usually partial differential equations) are smooth in the sense that the resulting Eulerian semi-derivative of the cost function is linear. Accordingly we speak of nonsmooth problems when the Eulerian semi-derivative is nonlinear. The nonlinearity can have two reasons: the first one is that the constraint itself is nonlinear, for instance it is a variational inequality of first or second kind; [19, 23, 22, 17]. The second and more obvious reason for the nonlinearity of the Eulerian semi-derivative is that the cost function itself is only directional differentiable which results in a nonlinearity of the Eulerian semi-derivative.

In this work we focus on nonsmooth cost functions in the aforementioned sense. To be more precise our cost function is maximum of a continuously differentiable function acting on continuous functions subject to a nonlinear PDE supplemented with mixed boundary conditions. This type of cost function can be used in various applications, such as mechanics, free boundary problems and electrical impedance tomography.

It is noteworthy that our approach has similarities to optimal control problems with pointwise state constraints; see [3]. We also refer to the work [5, 6, 12] for optimal control problems with  $L_\infty$  cost function. From the shape optimisation point of view our work is related to [13] where the square of the maximum norm subject to the (linear) Helmholtz equations was studied. The authors use the material derivative approach in conjunction with the notion of subgradient. Our results make use of the

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averaged adjoint approach [25] and the notion of Eulerian semi-derivative which allows the derivation of an optimality system under fairly general assumptions even with quasi-linear state equation.

A particularity of our approach, in contrast to previous ones [16, 2], is that we follow the paradigm first optimise-then-discretise. One main difficulty of our setting is that the partial differential equation is defined on a Lipschitz domain and supplemented with mixed boundary conditions for which no higher differentiability of the solution can be expected. In order to derive the Eulerian semi-differentiability we make use of maximal elliptic regularity results and combine them with the averaged adjoint approach from [25, 20]. Surprisingly also in this nonsmooth situation we can bypass the differentiation of the control-to-solution mapping by proving a weak Danskin-type theorem. The obtained Eulerian semi-derivative is then further studied in an infinite dimensional configuration by using valued reproducing kernel Hilbert spaces (vvRKHS). The effectiveness of vvRKHS for smooth shape optimisation problems has already been presented in [9]. This allows us to carry over results from the classical work [8].

## Structure of the paper

In Section 1, we recall basic facts from shape calculus and results on maximal elliptic regularity in dimension two.

In Section 2, we formulate the problem that is studied in the subsequent sections. We establish sensitivity results for a quasi-linear elliptic PDE with mixed boundary conditions. Furthermore the Eulerian semi-differentiability of a nonsmooth maximum-type cost is established using the averaged adjoint approach.

In Section 3, we study properties of the Eulerian semi-derivative and prove the existence of steepest descent directions and  $\epsilon$ -steepest descent directions. We then propose a discretisation of  $\epsilon$ -steepest descent directions adapted for finite elements.

In the final Section 4 we provide numerical experiments validating our theoretical findings. For that purpose a simple linear PDE with homogeneous Dirichlet boundary conditions is examined for which an analytical solution is available. We compare the results of the nonsmooth cost function with a  $L_2$ -type smooth cost function in order to highlight the difference.

# 1 Preliminaries

In this section, we recall some basics from shape calculus and PDE theory. For an in-depth treatment we refer the reader to the monographs [7, 24, 18, 15]. Numerous examples of PDE constrained shape functions and their shape derivatives can be found in [26].

## 1.1 Sobolev spaces and Gröger regular domains

We consider special subsets  $\Omega \subset \mathbf{R}^2$  satisfying the following conditions.

**Definition 1.1** ([14]). *Let  $\Omega \subset \mathbf{R}^2$  and  $\Gamma \subset \partial\Omega$  be given. We say that  $\Omega \cup \Gamma$  is regular (in the sense of Gröger) if  $\Omega$  is a bounded Lipschitz domain,  $\Gamma$  is a relatively open part of the boundary  $\partial\Omega$ ,  $\Gamma_0 := \partial\Omega \setminus \Gamma$  has positive measure and  $\Gamma_0$  is the finite union of closed and non-degenerated curved pieces of  $\partial\Omega$ .*

**Remark 1.2.** *For higher dimensions the previous definition can be extended via bi-lipschitz charts; cf. [14, Definition 2].*

With  $\Omega$ ,  $\Gamma$  and  $\Gamma_0$  defined as in Definition 1.1, we introduce for  $d \geq 1$ ,

$$\begin{aligned} C_c^\infty(\Omega, \mathbf{R}^d) &:= \{f|_\Omega : f \in C^\infty(\mathbf{R}^2, \mathbf{R}^d), \text{supp } f \cap \partial\Omega = \emptyset\}, \\ C_\Gamma^\infty(\Omega, \mathbf{R}^d) &:= \{f|_\Omega : f \in C^\infty(\mathbf{R}^2, \mathbf{R}^d), \text{supp } f \cap \Gamma_0 = \emptyset\}, \\ C_\Gamma(\Omega, \mathbf{R}^d) &:= \{f : f \in C(\bar{\Omega}, \mathbf{R}^d), f = 0 \text{ on } \Gamma_0\}. \end{aligned}$$

In the scalar valued case, that is,  $d = 1$ , we omit the last argument, for instance, we write  $C_c^\infty(\Omega) := C_c^\infty(\Omega, \mathbf{R}^1)$ . If we denote by  $\mathcal{M}(\Omega)$  the space of regular Borel measures, then by Riesz representation theorem  $\mathcal{M}(\bar{\Omega}) \simeq (C(\bar{\Omega}))^*$  and also  $\mathcal{M}(\Omega \cup \Gamma) \simeq (C_\Gamma(\Omega))^*$ .

For all finite integers  $p, p' \geq 1$  with  $1/p + 1/p' = 1$ , we define the Sobolev space

$$W_{\Gamma,p}^1(\Omega, \mathbf{R}^d) = \overline{C_\Gamma^\infty(\Omega, \mathbf{R}^d)}^{W_p^1}, \quad W_{\Gamma,p}^{-1}(\Omega, \mathbf{R}^d) := (W_{\Gamma,p'}^1(\Omega, \mathbf{R}^d))^*. \quad (1)$$

In case  $\Gamma = \emptyset$  we write  $\mathring{W}_p^1(\Omega, \mathbf{R}^d) := W_{\Gamma,p}^1(\Omega, \mathbf{R}^d)$ . In the scalar valued case we set  $W_{\Gamma,p}^1(\Omega) := W_{\Gamma,p}^1(\Omega, \mathbf{R}^1)$  and similarly for the other spaces. In case  $p = 2$  we use the notation  $W_{\Gamma,2}^1(\Omega, \mathbf{R}^d) =: H_\Gamma^1(\Omega, \mathbf{R}^d)$  and in case  $\Gamma = \emptyset$  also  $\mathring{H}^1(\Omega, \mathbf{R}^d) := W_{\Gamma,2}^1(\Omega, \mathbf{R}^d)$ .

## 1.2 Maximal elliptic regularity

Let  $\Omega$ ,  $\Gamma$  and  $\Gamma_0$  be as in Definition 1.1. Fix  $2 \leq q < \infty$  and denote by  $q'$  the conjugate of  $q$  defined by  $1/q + 1/q' = 1$ . Let  $b : \bar{\Omega} \times \mathbf{R}^3 \rightarrow \mathbf{R}^3$  be a function satisfying for all  $\eta, \theta \in \mathbf{R}^3$  and all  $x \in \bar{\Omega}$ :

$$\begin{aligned} b(\cdot, 0) &\in L_q(\Omega) \text{ and } b(\cdot, \eta) \text{ is measurable,} \\ (b(x, \eta) - b(x, \theta)) \cdot (\eta - \theta) &\geq m|\eta - \theta|^2, \quad m > 0, \\ |b(x, \eta) - b(x, \theta)| &\leq M|\eta - \theta|, \quad M > 0, \end{aligned} \quad (2)$$

where  $|\cdot|$  denotes the Euclidean norm. Notice that  $m \geq M$ . Let us denote  $Lu := \begin{pmatrix} u \\ \nabla u \end{pmatrix}$ . Then we define  $a(\cdot, \cdot)$  via  $a : W_{\Gamma,q}^1(\Omega) \times W_{\Gamma,q'}^1(\Omega) \rightarrow \mathbf{R}$ ,  $(v, w) \mapsto \int_\Omega b(x, Lv(x)) \cdot Lw(x) \, dx$  and the corresponding operator  $\mathcal{A}_q$ ,

$$\mathcal{A}_q : W_{\Gamma,q}^1(\Omega) \rightarrow W_{\Gamma,q}^{-1}(\Omega), \quad v \mapsto \mathcal{A}_q v := a(v, \cdot). \quad (3)$$

Let  $\mathcal{J}$  be defined by  $\langle \mathcal{J}u, v \rangle := \int_\Omega \nabla u \cdot \nabla v + uv \, dx$  for all  $u, v \in W_{\Gamma,2}^1(\Omega)$ . By Hölder's inequality it easily follows that  $\mathcal{J} : W_{\Gamma,p}^1(\Omega) \rightarrow W_{\Gamma,p}^{-1}(\Omega)$  is well-defined for all  $p \geq 2$ . With the help of the operator  $\mathcal{J}$  we may define  $M_p := \sup\{\|v\|_{W_p^1(\Omega)} : v \in W_{\Gamma,p}^1(\Omega), \|\mathcal{J}v\|_{W_{\Gamma,p}^{-1}} \leq 1\}$ . It is clear that  $M_2 = 1$ .

Henceforth it is useful to collect all regular domains:  $\Xi := \{(\Omega, \Gamma) : \Omega \subset \mathbf{R}^2, \Gamma \subset \partial\Omega, \text{ and } \Omega \cup \Gamma \text{ is regular}\}$ . We define  $\Omega^\Gamma := (\Omega, \Gamma)$ .

**Definition 1.3.** Denote by  $R_q$ ,  $2 \leq q < \infty$ , the set of regular domains  $\Omega^\Gamma \in \Xi$  for which  $\mathcal{J}$  maps  $W_{\Gamma,q}^1(\Omega)$  onto  $W_{\Gamma,q}^{-1}(\Omega)$ .

The following result is [11, Lemma 1].

**Lemma 1.4.** Let  $\Omega^\Gamma \in R_q$  for some  $q > 2$ . Then  $\Omega^\Gamma \in R_p$  for  $2 \leq p \leq q$  and  $M_q \leq M_p^\theta$  if  $\frac{1}{p} = \frac{(1-\theta)}{2} + \frac{\theta}{q}$ .

**Remark 1.5.** • If  $\Omega \subset \mathbf{R}^2$  is a bounded domain of class  $C^1$ , then  $(\Omega, \emptyset) \in \cap_{q \geq 2} R_q$ ; cf. [11, Remark 7].

- For every regular  $(\Omega, \Gamma) \in \Xi$  there is  $q > 2$ , so that  $\Omega^\Gamma \in R_q$ ; cf. [11, Theorem 3].
- If  $\Omega^\Gamma \in R_q$ , then  $M_q < \infty$ .

We can now state a result showing that the operator  $\mathcal{A}_q$  (in dimension two) is always an isomorphism for some (possibly small)  $q > 2$ . We recall the following version of [11, Theorem 1].

**Theorem 1.6** ([11]). Let  $\Omega^\Gamma \in R_{q_0}$ ,  $q_0 \geq 2$ . Suppose that  $b(\cdot, \cdot)$  satisfies Assumption 2 with  $q_0$  and let  $\mathcal{A}_q$  be defined by (3). Then  $\mathcal{A}_q : W_{\Gamma,q}^1(\Omega) \rightarrow W_{\Gamma,q}^{-1}(\Omega)$  is an isomorphism provided that  $q \in [2, q_0]$  and  $M_q k < 1$ , where  $k := (1 - m^2/M^2)^{1/2}$ . In that case

$$\|\mathcal{A}_q^{-1}f - \mathcal{A}_q^{-1}g\|_{W_q^1(\Omega)} \leq c_q \|f - g\|_{W_{\Gamma,q}^{-1}(\Omega)} \quad \text{for all } f, g \in W_{\Gamma,q}^{-1}(\Omega), \quad (4)$$

where  $c_q := mM^{-2}M_q(1 - M_qk)^{-1}$ . Finally,  $M_qk < 1$  is satisfied if

$$\frac{1}{q} > \frac{1}{2} - \left( \frac{1}{2} - \frac{1}{q_0} \right) \frac{|\log k|}{\log M_{q_0}}. \quad (5)$$

**Corollary 1.7.** *For small  $q > 2$  the constant  $c_q$  in (1.6) can be chosen to be independent of  $q$ .*

*Proof.* Assume first  $k = 0$ . Then Lemma 1.4 shows  $M_q \leq M_{q_0}^\theta$  with  $\frac{1}{q} = \frac{(1-\theta)}{2} + \frac{\theta}{q_0}$  (or  $\theta = \frac{q_0}{q} \frac{q-2}{q_0-2}$ ). Therefore  $c_q \leq mM^{-2}M_q \leq mM^{-2}M_{q_0}^\theta \leq mM^{-2} \max_{q \in [2, q_0]} M_{q_0}^{\theta(q)}$  and the maximum is attained as  $\theta(\cdot)$  is continuous on  $[2, q_0]$ .

Assume now  $k > 0$ . As shown in [11], inequality (5) follows from Lemma 1.4. To be more precise the estimate  $M_q \leq M_{q_0}^\theta$  with  $\frac{1}{q} = \frac{(1-\theta)}{2} + \frac{\theta}{q_0}$  shows that  $M_q^\theta k < 1$  implies  $M_{q_0}k < 1$  and indeed elementary computations show that  $M_{q_0}^\theta k < 1$  is equivalent to (5). In much the same way one can use Lemma 1.4 to show that  $M_qk \leq 1 - \epsilon$ , where  $\epsilon > 0$ , is satisfied if

$$\frac{1}{q} \geq \frac{1}{2} - \left( \frac{1}{2} - \frac{1}{q_0} \right) \left( \frac{\log(1 - \epsilon)}{\log M_{q_0}} + \frac{|\log k|}{\log M_{q_0}} \right). \quad (6)$$

In fact  $M_{q_0}^\theta k < 1 - \epsilon$  with  $\theta = \frac{q_0}{q} \frac{q-2}{q_0-2}$  is equivalent to (6). This shows that there is  $\epsilon > 0$  so that for all small  $q > 2$  we have  $c_q \leq mM^{-2}M_q(1 - M_qk)^{-1} \leq mM^{-2}(1 - \epsilon)/(k\epsilon)$  and thus  $c_q$  in (4) can be replaced by  $mM^{-2}(1 - \epsilon)/(k\epsilon)$  provided  $q > 2$  is small enough.  $\square$

### 1.3 Shape functions, shape derivative and shape gradients

Let  $D \subset \mathbf{R}^d$ ,  $d \geq 1$ , be an open and bounded set. Given a vector field  $X \in \mathring{C}^{0,1}(D, \mathbf{R}^d)$ , we denote by  $\Phi_t$  the flow of  $X$  (short  $X$ -flow) given by  $\Phi_t(x_0) := x(t)$ , where  $x(\cdot)$  solves

$$x'(t) = X(x(t)) \text{ in } (0, \tau), \quad x(0) = x_0. \quad (7)$$

The space  $\mathring{C}^{0,1}(D, \mathbf{R}^d)$  comprises all bounded and Lipschitz continuous functions on  $\bar{D}$  vanishing on  $\partial D$ . It is a closed subspace of  $C^{0,1}(D, \mathbf{R}^d)$ , the space of bounded Lipschitz continuous mapping defined on  $\bar{D}$ . Similarly we denote by  $\mathring{C}^k(D, \mathbf{R}^d)$  all function  $k$ -times differentiable function on  $D$  vanishing on  $\partial D$ . Note that by the chain rule (omitting the space variable  $x$ )  $(\partial(\Phi_t^{-1})) \circ \Phi_t = (\partial\Phi_t)^{-1} =: \partial\Phi_t^{-1}$ . We denote by  $\wp(D)$  the powerset of  $D$ . Let  $\Xi \subset \wp(D)$  be given.

**Definition 1.8.** (i) *A mapping  $J : \Xi \subset \wp(D) \rightarrow \mathbf{R}$  is called real shape function or shape function.*

(ii) *A mapping  $u : \Xi \subset \wp(D) \rightarrow \mathbf{R}^D$  with values in  $\mathbf{R}^D := \{f : D \rightarrow \mathbf{R}\}$ , is called abstract shape function. The set  $\Xi$  is referred to as admissible set.*

**Definition 1.9.** *Let  $J : \Xi \subset \wp(D) \rightarrow \mathbf{R}$  a shape function defined on subsets of  $D$ . Assume that  $\mathcal{H}(D, \mathbf{R}^d) \subset \mathring{C}^1(D, \mathbf{R}^d)$  is a subspace. Let  $\Omega \in \Xi$  and  $X \in \mathcal{H}(D, \mathbf{R}^d)$  be such that  $\Phi_t(\Omega) \in \Xi$  for all  $t > 0$  sufficiently small. Then the Eulerian semi-derivative of  $J$  at  $\Omega$  in direction  $X$  is defined by*

$$dJ(\Omega)(X) := \lim_{t \searrow 0} \frac{J(\Phi_t(\Omega)) - J(\Omega)}{t}. \quad (8)$$

We say that  $J$  is

- (i) Eulerian semi-differentiable at  $\Omega$  in  $\mathcal{H}(D, \mathbf{R}^d)$ , if  $dJ(\Omega)(X)$  exists for all  $X \in \mathcal{H}(D, \mathbf{R}^d)$ .
- (ii) shape differentiable at  $\Omega$  in  $\mathcal{H}(D, \mathbf{R}^d)$  if  $dJ(\Omega)(X)$  exists for all  $X \in \mathcal{H}(D, \mathbf{R}^d)$  and  $X \mapsto dJ(\Omega)(X)$  is linear and continuous.

Another auxiliary result that is frequently used is the following:

**Lemma 1.10.** *Let  $D \subseteq \mathbf{R}^d$  be open and bounded and suppose  $X \in \mathring{C}^1(D, \mathbf{R}^d)$ .*

(i) We have

$$\begin{aligned} \frac{\partial \Phi_t - I}{t} &\rightarrow \partial X & \text{and} & & \frac{\partial \Phi_t^{-1} - I}{t} &\rightarrow -\partial X & \text{strongly in } C(\bar{D}, \mathbf{R}^{d,d}) \\ \frac{\det(\partial \Phi_t) - 1}{t} &\rightarrow \operatorname{div}(X) & & & & & \text{strongly in } C(\bar{D}). \end{aligned}$$

(ii) For all open sets  $\Omega \subseteq D$  and all  $\varphi \in L_p(\Omega)$ ,  $1 \leq p < \infty$ , we have

$$\varphi \circ \Phi_t \rightarrow \varphi \quad \text{strongly in } L_p(\Omega). \quad (9)$$

Moreover, if  $\varphi \in W_p^1(\Omega)$ ,  $1 \leq p < \infty$ , then we have

$$\frac{\varphi \circ \Phi_t - \varphi}{t} \rightarrow \nabla \varphi \cdot X \quad \text{strongly in } L_p(\Omega). \quad (10)$$

Consider a function  $J : \Xi \subset \wp(D) \rightarrow \mathbf{R}$  that is shape differentiable at  $\Omega \in \Xi$  where  $D \subset \mathbf{R}^d$ . Suppose there is a Hilbert space  $\mathcal{H}(\mathcal{X}, \mathbf{R}^d)$  of functions from  $\mathcal{X} \subset D$  into  $\mathbf{R}^d$  and assume  $dJ(\Omega) \in \mathcal{H}(\mathcal{X}, \mathbf{R}^d)^*$ .

**Definition 1.11.** The gradient of  $J$  at  $\Omega$  with respect to the space  $\mathcal{H}(\mathcal{X}, \mathbf{R}^d)$  and the inner product  $(\cdot, \cdot)_{\mathcal{H}(\mathcal{X}, \mathbf{R}^d)}$ , denoted  $\nabla J(\Omega)$ , is defined by

$$dJ(\Omega)(\varphi) = (\nabla J(\Omega), \varphi)_{\mathcal{H}(\mathcal{X}, \mathbf{R}^d)} \quad \text{for all } \varphi \in \mathcal{H}(\mathcal{X}, \mathbf{R}^d). \quad (11)$$

We also call  $\nabla J(\Omega)$  the  $\mathcal{H}(\mathcal{X}, \mathbf{R}^d)$ -gradient of  $J$  at  $\Omega$ .

## 1.4 Projections in Hilbert spaces

Let us recall the following basic result on projections in Hilbert spaces.

**Lemma 1.12.** Let  $H$  be a real Hilbert space,  $K \subset H$  a closed and convex subset and  $x_0 \in H$ . For  $x^* \in K$  the following statements are equivalent:

- (i)  $\|x_0 - x^*\|_H = \inf_{x \in K} \|x - x_0\|_H$
- (ii)  $(x_0 - x^*, x - x^*)_H \leq 0 \quad \text{for all } x \in K$ .

Moreover, for each  $x_0 \in H$  there exists a unique element  $x^* \in K$  satisfying (i) (or equivalently (ii)).

*Proof.* See [27, Satz V.3.2, p.219 and Lemma V. 3.3, p.220].  $\square$

The previous lemma allows to define the projection mapping  $P_K : H \rightarrow K$  via  $P_K(x_0) := x^*$ . We also call  $x^* = P_K(x_0)$  the projection of  $x_0$  on  $K$ . Accordingly  $x^* := P_K(0)$  is the point in  $K$  that is closest to the origin 0 and (ii) reads  $(x^*, x^*)_H \leq (x^*, x)_H$  for all  $x \in K$ .

## 2 Maximum shape function subject to a quasi-linear PDE

This section is devoted to the derivation of the Eulerian semi-differentiability of a nonsmooth shape function subject to a quasi-linear partial differential equation.

## 2.1 Problem formulation and setting

Let us fix an open and bounded hold-all set  $D \subset \mathbf{R}^2$ . In this paper we study the maximum shape function

$$J_\infty(\Omega^\Gamma) := \max_{x \in \Omega} \Psi(x, u(\Omega^\Gamma, x)), \quad (12)$$

where  $\Omega^\Gamma = (\Omega, \Gamma)$  belongs to  $\Xi := \{(\Omega, \Gamma) : \Omega \subset D, \Gamma \subset \partial\Omega, \text{ and } \Omega \cup \Gamma \text{ is regular}\}$ , and  $u(\cdot) = u(\Omega^\Gamma, \cdot)$  solves (in a weak sense) the following quasi-linear PDE with mixed boundary conditions

$$-\operatorname{div}(\beta(|\nabla u|^2)\nabla u) + u = f \quad \text{in } \Omega, \quad (13)$$

$$u = 0 \quad \text{on } \partial\Omega \setminus \Gamma (=:\Gamma_0), \quad (14)$$

$$\partial_\nu u = 0 \quad \text{on } \Gamma. \quad (15)$$

As usual  $\partial_\nu u := \nabla u \cdot \nu$  is the normal derivative and  $\nu$  denotes the outward pointing unit normal vector along  $\partial\Omega$ . The functions  $\Psi$ ,  $f$ , and  $\beta$  are specified below.

Our first task is to prove the Eulerian semi-differentiability of  $J_\infty(\cdot)$  at sets  $\Omega^\Gamma$  belonging to the admissible set  $\Xi$ . To emphasise the dependency of  $u$  on  $\Omega^\Gamma$  we write  $u(\Omega^\Gamma, \cdot)$ , however, we drop the index  $\Omega^\Gamma$  whenever no confusion arises. In what follows it is convenient to introduce the shape function

$$j(\Omega^{\Gamma,y}) := \Psi(y, u(\Omega^\Gamma, y)), \quad (16)$$

depending on the shape variable  $\Omega^{\Gamma,y} := (\Omega, \Gamma, y) \in \Xi \times \bar{\Omega}$ .

To make sense of  $J_\infty(\Omega^\Gamma)$  it suffices to have  $u \in W_{\Gamma,q}^1(\Omega)$  with  $q > 2$  since in that case Sobolev's embedding implies  $u \in C_\Gamma(\Omega)$ . In order to obtain this higher integrability of  $u$  we make the following assumptions.

**Assumption 2.1.** *We require the function  $\beta : \mathbf{R} \rightarrow \mathbf{R}$  to satisfy the following conditions:*

1. *There exist constants  $\bar{\beta}, \underline{\beta} > 0$  such that  $\bar{\beta} \leq \beta(x) \leq \underline{\beta}$  for all  $x \in \mathbf{R}$ .*
2. *For all  $x, y \in \mathbf{R}$ , we have  $(\beta(x) - \beta(y))(x - y) \geq 0$ .*
3. *The function  $\beta$  is continuously differentiable, that is,  $\beta \in C^1(\mathbf{R})$ .*
4. *There are constants  $k, K > 0$ , such that*

$$k|\eta|^2 \leq \beta(|p|^2)|\eta|^2 + 2\beta'(|p|^2)|p \cdot \eta|^2 \leq K|\eta|^2 \quad \text{for all } \eta, p \in \mathbf{R}^2. \quad (17)$$

**Remark 2.2.** *Notice that using (1) and (2) of the previous assumption, we obtain*

$$\underbrace{\beta(|p|^2)}_{\geq \bar{\beta}}|\eta|^2 + 2\underbrace{\beta'(|p|^2)}_{\geq 0}|p \cdot \eta|^2 \geq \bar{\beta}|\eta|^2 \quad \text{for all } \eta, p \in \mathbf{R}^2. \quad (18)$$

*So (1) and (2) imply the left inequality in item 4.*

**Assumption 2.3.** *We assume that  $f \in L_q(D)$  for some  $q > 2$ .*

**Assumption 2.4.** *We assume that the functions  $\Psi : \mathbf{R}^2 \times \mathbf{R} \rightarrow \mathbf{R}$  satisfies,*

- *for all  $x \in \mathbf{R}^2$ ,  $\Psi(x, \cdot) \in C^1(\mathbf{R})$  and  $\partial_\zeta \Psi \in C(\mathbf{R}^3)$ ,*
- *for all  $\zeta \in \mathbf{R}$ ,  $\Psi(\cdot, \zeta) \in C^1(\mathbf{R}^2)$ .*

**Example 2.5.** *A typical example of  $\Psi$  is the function  $\Psi(x, z) := |z - u_d(x)|^2$ , where  $u_d : \mathbf{R}^2 \rightarrow \mathbf{R}$  is some continuously differentiable function. For this choice of cost function we present numerical results in Section 4.*

**Lemma 2.6.** *Let  $\beta : \mathbf{R} \rightarrow \mathbf{R}$  satisfy Assumption 2.1. Then for all  $\theta, \eta \in \mathbf{R}^2$ ,*

$$k|\eta - \theta|^2 \leq (\beta(|\eta|^2)\eta - \beta(|\theta|^2)\theta) \cdot (\eta - \theta), \quad (19)$$

$$K|\eta - \theta| \geq |\beta(|\eta|^2)\eta - \beta(|\theta|^2)\theta|. \quad (20)$$

*Proof.* We obtain by the fundamental theorem of calculus,

$$\begin{aligned} (\beta(|\eta|^2)\eta - \beta(|\theta|^2)\theta) \cdot (\eta - \theta) &= \int_0^1 2\beta'(|s\eta + (1-s)\theta|^2)|(\eta - \theta) \cdot (s\eta + (1-s)\theta)|^2 ds \\ &\quad + \int_0^1 \beta(|s\eta + (1-s)\theta|^2)|\eta - \theta|^2 ds \quad \text{for all } \theta, \eta \in \mathbf{R}^2. \end{aligned} \quad (21)$$

Hence (19) follows from Assumption 2.1, item 4. The continuity (20) follows in the same way.  $\square$

Note that, in general,  $u \notin H^2(\Omega)$  due to the mixed boundary conditions. However, we have the following result.

**Lemma 2.7.** *Let Assumption 2.1 be satisfied and assume  $\Omega^\Gamma \in \Xi$ . For every small enough  $q > 2$ , there is a unique  $u \in W_{\Gamma,q}^1(\Omega)$  satisfying*

$$\int_{\Omega} \beta(|\nabla u|^2) \nabla u \cdot \nabla \varphi + u \varphi \, dx = \int_{\Omega} f \varphi \, dx \quad \text{for all } \varphi \in W_{\Gamma,q'}^1(\Omega) \quad (22)$$

or equivalently

$$\int_{\Omega} a(x, Lu(x)) \cdot L\varphi(x) \, dx = \int_{\Omega} f \varphi \, dx \quad \text{for all } \varphi \in W_{\Gamma,q'}^1(\Omega), \quad (23)$$

where

$$a(x, \zeta) := \begin{pmatrix} \zeta_0 \\ \beta(|\hat{\zeta}|^2)\hat{\zeta} \end{pmatrix}, \quad \zeta = \begin{pmatrix} \zeta_0 \\ \hat{\zeta} \end{pmatrix} \in \mathbf{R}^3, \quad Lu := \begin{pmatrix} u \\ \nabla u \end{pmatrix}. \quad (24)$$

*Proof.* We apply Theorem 1.6 to  $b(x, \zeta) := a(x, \zeta)$  with  $a$  defined in (24). We need to check the conditions stated in (2). It is clear that  $b(\cdot, 0) \in L_{\infty}(\Omega)$ . Since Assumption 2.1 is satisfied, Lemma 2.6 yields (19) and (20) and hence this implies the continuity and monotonicity properties for  $b(x, \cdot)$  stated in (2). Setting  $l = (1 - (m/M)^2)^{1/2}$  and  $m := \min\{k, 1\}$ ,  $M := \max\{K, 1\}$  we see that the condition  $M_q l < 1$  is satisfied provided  $q > 2$  is small enough (cf. (5)). So the result follows from Theorem 1.6.  $\square$

## 2.2 Analysis of the perturbed state equation

Let  $\Omega^\Gamma \subset \Xi$  be fixed and pick a vector field  $X \in \mathring{C}^1(D, \mathbf{R}^2)$  with associated  $X$ -flow  $\Phi_t$ . We set  $\Omega_t := \Phi_t(\Omega)$ ,  $t \geq 0$ , and consider (22) on the perturbed domain  $\Omega_t$  and perform a change of variables to obtain,

$$\int_{\Omega} \beta(|B(t)\nabla u^t|^2) A(t) \nabla u^t \cdot \nabla \varphi + \xi(t) u^t \varphi \, dx = \int_{\Omega} f^t \varphi \, dx \quad \text{for all } \varphi \in W_{\Gamma,q'}^1(\Omega), \quad (25)$$

where  $q \geq 2$  with its conjugate  $q' = q/(q-1)$ , and

$$A(t) := \det(\partial\Phi_t) \partial\Phi_t^{-1} \partial\Phi_t^{-\top}, \quad B(t) := \partial\Phi_t^{-\top}, \quad f^t := \det(\partial\Phi_t) f \circ \Phi_t, \quad \xi(t) := \det(\partial\Phi_t). \quad (26)$$

The existence and uniqueness of a solution of (25) is addressed below. It is convenient to rewrite (25) as

$$\int_{\Omega} a^t(x, Lu^t(x)) \cdot L\varphi(x) \, dx = \int_{\Omega} f \varphi \, dx \quad \text{for all } \varphi \in W_{\Gamma,q}^1(\Omega) \quad (27)$$

with the definition

$$a^t(x, \zeta) := \begin{pmatrix} \xi(t, x) \zeta_0 \\ \beta(|B(t, x)\hat{\zeta}|^2) A(t, x) \hat{\zeta} \end{pmatrix}, \quad \zeta = \begin{pmatrix} \zeta_0 \\ \hat{\zeta} \end{pmatrix} \in \mathbf{R}^3. \quad (28)$$

We associate with  $a^t$  the operator

$$\mathcal{A}_q^t : W_{\Gamma,q}^1(\Omega) \rightarrow W_{\Gamma,q}^{-1}(\Omega), \quad \langle \mathcal{A}_q^t v, w \rangle := \int_{\Omega} a^t(x, Lv(x)) \cdot Lw(x) \, dx, \quad (29)$$

where  $q > 2$ . We show next that for all sufficiently small  $q > 2$  and  $t > 0$  the operators  $\mathcal{A}_q^t$  are isomorphisms from  $W_{\Gamma,q}^1(\Omega)$  onto  $W_{\Gamma,q}^{-1}(\Omega)$ . The main task is to show that  $q$  is independent of  $t$  provided it is small enough. We begin with the following lemma.

**Lemma 2.8.** *For every  $\epsilon > 0$ , there exists  $\delta > 0$ , so that,*

$$A(t, x)\eta \cdot \eta \geq (1 - \epsilon)|\eta|^2 \quad \text{for all } \eta \in \mathbf{R}^d, \text{ for all } (t, x) \in [0, \delta] \times \bar{D}, \quad (30)$$

$$\|A(t)\|_{C(\bar{D}, \mathbf{R}^{d,d})} \leq 1 + \epsilon \quad \text{for all } t \in [0, \delta], \quad (31)$$

$$1 - \epsilon \leq |B(t, x)\eta| \leq 1 + \epsilon \quad \text{for all } \eta \in \mathbf{R}^d \text{ for all } (t, x) \in [0, \delta] \times \bar{D}, \quad (32)$$

$$1 - \epsilon \leq \xi(t, x) \leq 1 + \epsilon \quad \text{for all } (t, x) \in [0, \delta] \times \bar{D}. \quad (33)$$

*Proof.* We only prove (32) as the other estimates can be shown in much the same way. Since  $B : [0, \tau] \rightarrow C(\bar{D}, \mathbf{R}^{d,d})$  is continuous and  $B(0) = I$ , we find for every  $\epsilon > 0$  a number  $\delta > 0$  so that  $\|B(t) - I\|_{C(\bar{D}, \mathbf{R}^{d,d})} \leq \epsilon$  for all  $|t| \leq \delta$ . Hence the left inequality in (32) follows by the reverse triangle inequality. As for the right inequality in (32) note that for all  $\eta \in \mathbf{R}^d$  and all  $(t, x) \in [0, \delta] \times \bar{D}$ ,

$$\begin{aligned} |\eta|^2 &= |(I - B(t, x)) \cdot \eta| + |B(t, x)\eta| \\ &\leq \underbrace{\|B(t) - I\|_{C(\bar{D}, \mathbf{R}^{d,d})}}_{\leq \epsilon} |\eta|^2 + |B(t, x)\eta| \end{aligned} \quad (34)$$

which is equivalent to (32).  $\square$

**Lemma 2.9.** *For each  $\Omega^\Gamma \in \Xi$ , there exist  $q_0 > 2$  and  $\delta > 0$ , so that for all  $t \in [0, \delta]$  and all  $q \in [2, q_0]$  the mapping  $\mathcal{A}_q^t : W_{\Gamma, q}^1(\Omega) \rightarrow W_{\Gamma, q}^{-1}(\Omega)$  is an isomorphism. Moreover, there is a constant  $c > 0$  independent of  $t$ , so that*

$$\|(\mathcal{A}_q^t)^{-1}f - (\mathcal{A}_q^t)^{-1}g\|_{W_q^1(\Omega)} \leq c\|f - g\|_{W_{\Gamma, q}^{-1}(\Omega)} \quad \text{for all } f, g \in W_{\Gamma, q}^{-1}(\Omega) \quad (35)$$

for all  $t \in [0, \delta]$ .

*Proof.* According to Theorem 1.6 and Lemma 2.7 there is  $q_0 > 2$  so that  $\mathcal{A}(\cdot) = -\operatorname{div}(\beta(|\nabla \cdot|^2)\nabla \cdot)$  is an isomorphism from  $W_{\Gamma, q}^1(\Omega)$  onto  $W_{\Gamma, q}^{-1}(\Omega)$  for all  $q \in (2, q_0]$ . Indeed setting  $m := \min\{k, 1\}$  and  $M := \max\{K, 1\}$  we get  $M_q(1 - m^2/M^2)^{1/2} < 1$  provided  $q$  is close enough to 2 (cf. (5)). Similarly to (21), we can write

$$\begin{aligned} &(\beta(|B(t)\eta|^2)A(t)\eta - \beta(|B(t)\theta|^2)A(t)\theta) \cdot (\eta - \theta) \\ &= \int_0^1 \xi(t)2\beta'(|B(t)(s\eta + (1-s)\theta)|^2)|B(t)(\eta - \theta) \cdot B(t)(s\eta + (1-s)\theta)|^2 ds \\ &+ \int_0^1 \xi(t)\beta(|B(t)(s\eta + (1-s)\theta)|^2)|B(t)(\eta - \theta)|^2 ds \quad \text{for all } \theta, \eta \in \mathbf{R}^2, \text{ for all } t. \end{aligned} \quad (36)$$

So using Assumption 2.1 and Lemma 2.8, we get

$$(\beta(|B(t)\eta|^2)A(t)\eta - \beta(|B(t)\theta|^2)A(t)\theta) \cdot (\eta - \theta) \geq (m - \epsilon)|B(t)(\eta - \theta)|^2 \geq (k - \epsilon)|\eta - \theta|^2 \quad (37)$$

for all  $\theta, \eta \in \mathbf{R}^2$  and all sufficiently small  $t$ . In a similar manner we can show

$$|\beta(|B(t)\eta|^2)A(t)\eta - \beta(|B(t)\theta|^2)A(t)\theta| \leq (M + \epsilon)|\eta - \theta| \quad (38)$$

for all  $\theta, \eta \in \mathbf{R}^2$  and all sufficiently small  $t$ . This implies that we find for  $\epsilon > 0$  a number  $\delta > 0$  so that

$$(a^t(x, \eta) - a^t(x, \theta)) \cdot (\eta - \theta) \geq (m - \epsilon)|\eta - \theta|^2, \quad (39)$$

$$|a^t(x, \eta) - a^t(x, \theta)| \leq (M + \epsilon)|\eta - \theta| \quad (40)$$

for all  $t \in [0, \delta]$  and for all  $\theta, \eta \in \mathbf{R}^3$ . Noting that  $a^t(\cdot, 0) \in L_\infty(D)$  we can apply again Theorem 1.6 and obtain that  $\mathcal{A}_q^t$  is in fact an isomorphism when we choose  $\epsilon$  so small that  $M_q(1 - (m - \epsilon)^2/(M + \epsilon)^2)^{1/2} < 1$  which is possible since  $\lim_{\epsilon \searrow 0} (m - \epsilon)/(M + \epsilon) = m/M$  and  $M_q(1 - m^2/M^2)^{1/2} < 1$ .  $\square$

**Definition 2.10.** *For  $\Omega^\Gamma \in \Xi$  we define  $q_0 > 2$  to be a number as in Lemma 2.9.*



**Corollary 2.11.** Suppose that  $X \in \dot{C}^1(D, \mathbf{R}^2)$ ,  $\Omega^\Gamma \in \Xi$  and  $q \in (2, q_0]$ .

(a) If  $f \in L_q(\Omega)$ , then the family of solutions  $\{u^t\}$  of (25) satisfies

$$\lim_{t \searrow 0} \|u^t - u\|_{W_q^1(\Omega)} = 0 \quad \text{and} \quad \lim_{t \searrow 0} \|u^t - u\|_{C(\bar{\Omega})} = 0. \quad (41)$$

(b) If  $f \in W_q^1(\Omega)$ , then there is  $\tau > 0$  and  $c > 0$ , so that  $\{u^t\}$  satisfies

$$\|u^t - u\|_{C(\bar{\Omega})} + \|u^t - u\|_{W_q^1(\Omega)} \leq ct \quad \forall t \in [0, \tau]. \quad (42)$$

*Proof.* Let us first show (a). By Lemma 2.9 we find  $\delta > 0$  and  $q_0 > 2$ , so that  $\|u^t\|_{W_q^1(\Omega)} = \|(\mathcal{A}_q^t)^{-1} f^t\|_{W_q^1(\Omega)} \leq c \|f^t\|_{W_{\Gamma, q}^{-1}(\Omega)}$  for all  $q \in (2, q_0]$  and all  $t \in [0, \delta]$  and using Hölder's inequality the right hand side can be further estimated

$$\|f^t\|_{W_{\Gamma, q}^{-1}(\Omega)} = \sup_{\substack{\varphi \in W_{\Gamma, q'}^1(\Omega) \\ \|\varphi\|_{W_{q'}^1(\Omega)} \leq 1}} \left| \int_{\Omega} f^t \varphi \, dx \right| \leq \|f^t\|_{L_q(\Omega)}. \quad (43)$$

The boundedness of  $\|f^t\|_{L_q(\Omega)}$  follows from Lemma 1.10. So  $u^t$  is bounded in  $W_{\Gamma, q}^1(\Omega)$  with  $q \in (2, q_0]$ . Now by definition  $u^t$  and  $u := u^0$  satisfy (setting  $\mathcal{A}_q := \mathcal{A}_q^0$ ) the operator equations  $\mathcal{A}_q^t u^t = f^t$  and  $\mathcal{A}_q u = f$ . Therefore the difference  $z^t := u^t - u$  solves  $\mathcal{A}_q z^t = -(\mathcal{A}_q^t - \mathcal{A}_q)u^t - (f^t - f) \in W_{\Gamma, q}^{-1}(\Omega)$  and hence using again Lemma 2.9 we find  $c > 0$  so that for all  $t$ ,

$$\begin{aligned} \|z^t\|_{W_q^1(\Omega)} &\leq c \| -(\mathcal{A}_q^t - \mathcal{A}_q)u^t - (f^t - f) \|_{W_{\Gamma, q}^{-1}(\Omega)} \\ &\leq c (\|(\mathcal{A}_q^t - \mathcal{A}_q)u^t\|_{W_{\Gamma, q}^{-1}(\Omega)} + \|f^t - f\|_{W_{\Gamma, q}^{-1}(\Omega)}). \end{aligned} \quad (44)$$

Furthermore we have for a.e.  $x \in \Omega$  and all  $t \in [0, \delta]$ ,

$$\begin{aligned} &|\beta(|B(t, x) \nabla u^t(x)|^2) A(t, x) \nabla u^t(x) - \beta(|\nabla u^t(x)|^2) \nabla u^t(x)| \leq \\ &\quad \underbrace{|A(t, x) - \xi(t, x) B^\top(t, x)|}_{\leq ct, \text{ by Lemma 1.10, (i)}} \underbrace{|\beta(|B(t, x) \nabla u^t(x)|^2) \nabla u^t(x)|}_{\leq c |\nabla u^t(x)|, \text{ by Lemma 2.8}} \\ &\quad + \underbrace{|\beta(|B(t, x) \nabla u^t(x)|^2) B(t, x) \nabla u^t(x) - \beta(|\nabla u^t(x)|^2) \nabla u^t(x)|}_{\leq K |B(t, x) - I| |\nabla u^t(x)|, \text{ by (20)}} \leq ct |\nabla u^t(x)|. \end{aligned} \quad (45)$$

So using again Hölder's inequality yields

$$\begin{aligned} &\|(\mathcal{A}_q^t - \mathcal{A}_q)u^t\|_{W_{\Gamma, q}^{-1}(\Omega)} \\ &= \sup_{\substack{\varphi \in W_{\Gamma, q'}^1(\Omega) \\ \|\varphi\|_{W_{q'}^1(\Omega)} \leq 1}} \left| \int_{\Omega} \underbrace{(\beta(|B(t) \nabla u^t|^2) A(t) \nabla u^t - \beta(|\nabla u^t|^2) \nabla u^t) \cdot \nabla \varphi}_{\leq ct |\nabla u^t| |\nabla \varphi|, \text{ by (45)}} + \underbrace{(\xi(t) - 1)}_{\leq ct, \text{ by Lemma 1.10, (i)}} u^t \varphi \, dx \right| \\ &\leq ct \left( \sup_{\substack{\varphi \in W_{\Gamma, q'}^1(\Omega) \\ \|\varphi\|_{W_{q'}^1(\Omega)} \leq 1}} \int_{\Omega} |\nabla u^t| |\nabla \varphi| \, dx + \sup_{\substack{\varphi \in W_{\Gamma, q'}^1(\Omega) \\ \|\varphi\|_{W_{q'}^1(\Omega)} \leq 1}} \int_{\Omega} |u^t \varphi| \, dx \right) \leq ct \underbrace{\|u^t\|_{W_q^1(\Omega)}}_{\leq c} \leq ct \end{aligned} \quad (46)$$

and similarly

$$\|f^t - f\|_{W_{\Gamma, q}^{-1}(\Omega)} \leq \underbrace{\|f^t - f\|_{L_q(\Omega)}}_{=o(1), \text{ Lemma 1.10, (ii)}}. \quad (47)$$

Now using (46) and (47) to estimate the right hand side of (44) yields  $\lim_{t \searrow 0} \|u^t - u\|_{W_q^1(\Omega)} = 0$ . Since  $q > 2$  the space  $W_{\Gamma, q}^1(\Omega)$  embeds continuously into  $C_\Gamma(\Omega)$  and we obtain  $\lim_{t \searrow 0} \|u^t - u\|_{C(\bar{\Omega})} = 0$ .

Finally item (b) follows since for  $f \in W_q^1(D)$ ,  $q > 2$ , we obtain the estimate  $\|f^t - f\|_{L_q(D)} \leq ct$  (cf. Lemma 1.10, (ii)). This finishes the proof.  $\square$

### 2.3 Analysis of the averaged adjoint state equation

At first we introduce for fixed  $y \in \bar{\Omega}$  and  $t \geq 0$  the Lagrangian function:

$$G_y(t, v, w) := \Psi(\Phi_t(y), v(y)) + \int_{\Omega} \beta(|B(t)\nabla v|^2) A(t) \nabla v \cdot \nabla w + \xi(t)vw - f^t w \, dx, \quad (48)$$

where  $v \in W_q^1(\Omega)$  and  $w \in W_{q'}^1(\Omega)$  with  $q > 2$  and  $q' := q/(q-1)$ . Notice that  $G_y = G_y^X$  also depends on the vector field  $X$ , however, to keep the notation simple we omit this dependency. In the rest of the paper we assume  $f \in W_q^1(D)$ .

**Definition 2.12.** Let  $y \in \bar{\Omega}$  be fixed and  $q \in (2, q_0]$ , where  $q_0$  is as in Lemma 2.9. We introduce the averaged adjoint equation as:

$$\text{Find } p_y^t \in W_{\Gamma, q'}^1(\Omega), \quad \int_0^1 d_v G_y(t, su^t + (1-s)u, p_y^t)(\varphi) \, ds = 0 \quad \text{for all } \varphi \in W_{\Gamma, q}^1(\Omega). \quad (49)$$

The function  $p_y^t$  is referred to as averaged adjoint state.

The reason for introducing the averaged adjoint equation is the following identity

$$G_y(t, u^t, p_y^t) - G_y(t, u, p_y^t) = \int_0^1 d_v G_y(t, su^t + (1-s)u, p_y^t)(u^t - u) \, ds = 0, \quad (50)$$

where the last equality follows in view of (49) and  $u^t - u \in W_{\Gamma, q}^1(\Omega)$ . Now with the Lagrangian  $G_y$  the shape functions  $J_{\infty}(\cdot)$  and  $j(\cdot)$  can be expressed as

$$J_{\infty}(\Omega_t^{\Gamma}) = \max_{y \in \bar{\Omega}} j(\Omega_t^{\Gamma, y}), \quad j(\Omega_t^{\Gamma, y}) = G_y(t, u, p_y^t), \quad y \in \bar{\Omega}. \quad (51)$$

Consequently it suffice to study the differentiability of  $t \mapsto \max_{y \in \bar{\Omega}} G_y(t, u, p_y^t)$  and  $t \mapsto G_y^X(t, u, p_y^t)$  in order to prove that  $J_{\infty}(\cdot)$  is Eulerian semi-differentiable at  $\Omega^{\Gamma} \in \Xi$  and  $j(\cdot)$  is shape differentiable at all  $\Omega^{\Gamma, y}$ , where  $\Omega^{\Gamma} \in \Xi$  and  $y \in \bar{\Omega}$ . This is the content of the following two sections. At first we study the averaged adjoint equation. We notice that (49) is equivalent to

$$\int_{\Omega} b^t(x, u^t, u) L p_y^t \cdot L \varphi \, dx = -\bar{\Psi}^t(y, u^t, u) \varphi(y) \quad \text{for all } \varphi \in W_{\Gamma, q}^1(\Omega), \quad (52)$$

where

$$b^t(x, u^t, u) := \int_0^1 \partial_{\zeta} a^t(x, s L u^t(x) + (1-s) L u(x)) \, ds, \quad (53)$$

$$\bar{\Psi}^t(y, u^t(y), u(y)) := \int_0^1 \partial_{\zeta} \Psi(\Phi_t(y), s u^t(y) + (1-s) u(y)) \, ds. \quad (54)$$

In view of

$$\partial_{\zeta} a^t(x, \zeta) = \begin{pmatrix} \xi(t, x) \zeta_0 \\ \beta(|B(t, x) \hat{\zeta}|^2) A(t, x) + 2\beta'(|B(t, x) \hat{\zeta}|^2) A(t, x) \hat{\zeta} \otimes B(t, x) \hat{\zeta} \end{pmatrix}, \quad \zeta = \begin{pmatrix} \zeta_0 \\ \hat{\zeta} \end{pmatrix}, \quad (55)$$

it immediately follows from Assumption 2.1, item 4, that there is a constant  $c > 0$  so that  $\|b^t(\cdot, u^t, u)\|_{L_{\infty}(\Omega)} \leq c$  for all  $t$ . Notice that at  $t = 0$  equation (52) reduces to the usual adjoint state equation:

$$\text{find } p_y \in W_{q'}^1(\Omega), \quad \int_{\Omega} b(x, u) L p_y \cdot L \varphi \, dx = -\partial_u \Psi(y, u(y)) \varphi(y) \quad \text{for all } \varphi \in W_{\Gamma, q}^1(\Omega), \quad (56)$$

where  $b(x, \cdot) := b^0(x, \cdot, \cdot)$ . We associate with  $b^t$  the (linear) operator  $\mathcal{B}_{q'}^t : W_{\Gamma, q'}^1(\Omega) \rightarrow W_{\Gamma, q'}^{-1}(\Omega)$  defined by  $\langle \mathcal{B}_{q'}^t v, w \rangle := \int_{\Omega} b^t(x, u^t, u) L v \cdot L w \, dx$ .

The proof of the following lemma follows [21].

**Lemma 2.13.** *Let  $\Omega^\Gamma \in \Xi$  with associated  $q_0 > 2$  be given. Then there exists  $\delta > 0$ , so that the averaged operator  $\mathcal{B}_{q'}^t : W_{\Gamma,q'}^1(\Omega) \rightarrow W_{\Gamma,q'}^{-1}(\Omega)$  is an isomorphism for all  $t \in [0, \tau]$ . Moreover, there is a constant  $c > 0$ , so that for all  $t \in [0, \delta]$ ,*

$$\|(\mathcal{B}_{q'}^t)^{-1}f - (\mathcal{B}_{q'}^t)^{-1}g\|_{W_{\Gamma,q'}^{-1}(\Omega)} \leq c\|f - g\|_{W_{\Gamma,q'}^{-1}(\Omega)} \quad \text{for all } f, g \in W_{\Gamma,q'}^1(\Omega). \quad (57)$$

*Proof.* Let  $\epsilon > 0$  be fixed. Using Assumption 2.1 it is readily checked that there is  $\delta > 0$  so that for all  $t \in [0, \delta]$  the function  $b^t(x, \zeta) := b^t(x, u^t, u)\zeta$  satisfies (2) with  $m = \min\{1, k\} - \epsilon$  and  $M = \max\{K, 1\} + \epsilon$  for  $t$  sufficiently small. Hence there is  $\delta > 0$  so that the mapping  $\mathcal{B}_q^t : W_{\Gamma,q}^1(\Omega) \rightarrow W_{\Gamma,q}^{-1}(\Omega)$  is an isomorphism for all  $t \in [0, \delta]$ . Thus by the closed range theorem also the adjoint  $(\mathcal{B}_q^t)^* = \mathcal{B}_{q'}^t : W_{\Gamma,q'}^1(\Omega) \rightarrow W_{\Gamma,q'}^{-1}(\Omega)$  is an isomorphism with continuous inverse and we finish the proof.  $\square$

**Lemma 2.14.** *Let  $\Omega^\Gamma \in \Xi$  with associated  $q_0 > 2$  be given. Assume  $y_t : \mathbf{R} \rightarrow \mathbf{R}^2$  is a function that is continuous from the right in  $t = 0$  with  $y(0) = y \in \Omega$ . For  $t \geq 0$  we denote by  $p_{y_t} \in W_{\Gamma,q'}^1(\Omega)$  the solution of*

$$\int_{\Omega} b^t(x, u^t, u) L p_{y_t}^t \cdot L \varphi \, dx = -\bar{\Psi}^t(y_t, u^t(y_t), u(y_t)) \varphi(y_t) \quad \text{for all } \varphi \in W_{\Gamma,q}^1(\Omega), \quad (58)$$

where  $q > 2$  is the conjugate of  $q'$ , that is,  $1/q' + 1/q = 1$ . Then  $1 < q' < 2$  and we get  $p_{y_t}^t \rightharpoonup p_y$  weakly in  $W_{\Gamma,q'}^1(\Omega)$ , where  $p_y$  denotes the solution of (56).

*Proof.* By Sobolev's embedding the inclusion mapping  $E_\Gamma : W_{\Gamma,q}^1(\Omega) \rightarrow C_\Gamma(\Omega)$  is continuous for all  $q > 2$ . Thus the adjoint  $E_\Gamma^* : C_\Gamma(\Omega) \rightarrow W_{\Gamma,q}^1(\Omega)$  is continuous, too. As the mapping  $\alpha^t \delta_{y_t} : C_\Gamma(\Omega) \rightarrow \mathbf{R}$ ,  $f \mapsto \alpha^t f(y_t)$ , where  $\alpha^t := \bar{\Psi}^t(y_t, u^t(y_t), u(y_t)) \in \mathbf{R}$ , is continuous, we can rewrite (58) as

$$\langle \mathcal{B}_{q'}^t p_{y_t}^t, \varphi \rangle_{W_{\Gamma,q'}^1, W_{\Gamma,q'}^{-1}} = -\langle E_\Gamma^*(\alpha^t \delta_{y_t}), \varphi \rangle_{W_{\Gamma,q'}^1, W_{\Gamma,q'}^{-1}} \quad \text{for all } \varphi \in W_{\Gamma,q}^1(\Omega). \quad (59)$$

Now applying Lemma 2.14 yields  $\|p_{y_t}^t\|_{W_{\Gamma,q'}^1(\Omega)} \leq c\|E_\Gamma^*(\alpha^t \delta_{y_t})\|_{W_{\Gamma,q'}^{-1}(\Omega)} \leq c\alpha^t \|\delta_{y_t}\|_{(C_\Gamma(\Omega))^*} \leq c$  for all  $t \in [0, \delta]$ . So for each real nullsequence  $(t_n)$  there is a subsequence and  $z \in W_{\Gamma,q'}^1(\Omega)$ , still indexed the same, such that  $p_{y_{t_n}} \rightharpoonup z$  in  $W_{\Gamma,q'}^1(\Omega)$ . Therefore passing to the limit in (58), we conclude by uniqueness of the adjoint state equation that  $z = p_y$ . This also shows  $p_{y_t} \rightharpoonup p_y$  in  $W_{\Gamma,q'}^1(\Omega)$  as  $t \searrow 0$ .  $\square$

## 2.4 Shape derivative of $j(\cdot)$ via averaged adjoint

Let  $X \in \mathring{C}^1(D, \mathbf{R}^2)$  be a given vector field and  $\Phi_t$  the corresponding flow. Let  $\Omega^\Gamma \in \Xi$  and  $y \in \bar{\Omega}$ . Then the perturbation of the set  $\Omega^{\Gamma,y} = (\Omega^\Gamma, y)$  is defined by  $\Omega_t^{\Gamma,y} := (\Omega_t, \Gamma_t, y_t)$ , where  $\Omega_t := \Phi_t(\Omega)$ ,  $\Gamma_t := \Phi_t(\Gamma)$  and  $y_t := \Phi_t(y)$ . The Eulerian semi-derivative of  $j(\cdot)$  at  $\Omega^{\Gamma,y}$  in direction  $X$  is then defined by  $dj(\Omega^{\Gamma,y})(X) = \lim_{t \searrow 0} (j(\Omega_t^{\Gamma,y}) - j(\Omega^{\Gamma,y}))/t$ . Let us now prove that  $j(\cdot)$  is in fact shape differentiable.

**Theorem 2.15.** *Let  $\Omega^\Gamma \in \Xi$  and  $y \in \bar{\Omega}$  be given and assume  $2 < q < q_0$ . The shape function  $j(\cdot)$  is shape differentiable at every  $\Omega^{\Gamma,y}$  and the derivative in direction  $X \in \mathring{C}^1(D, \mathbf{R}^2)$  is given by*

$$dj(\Omega^{\Gamma,y})(X) = \partial_t G_y^X(0, u, p_y), \quad (60)$$

where  $(u, p_y) \in W_{\Gamma,q}^1(\Omega) \times W_{\Gamma,q'}^1(\Omega)$  solves (22) and (56), respectively.

It is sufficient to prove the following lemma.

**Lemma 2.16.** *Let  $\Omega^\Gamma \in \Xi$  and  $y \in \bar{\Omega}$  be given and assume  $2 < q < q_0$ . For all functions  $y_t = y(t) : \mathbf{R} \rightarrow \mathbf{R}^2$  that are continuous from the right in  $t = 0$ , we have*

$$\lim_{t \searrow 0} \frac{G_{y_t}(t, u^t, p_{y_t}^t) - G_{y_t}(0, u, p_{y_t})}{t} = \partial_t G_y(0, u, p_y), \quad (61)$$

where  $(u, p_y) \in W_{\Gamma,q}^1(\Omega) \times W_{\Gamma,q'}^1(\Omega)$  solves (22) and (56), respectively.

*Proof.* By definition of the function  $p_{y_t}^t$ ,  $t > 0$ , we get (cf. (50))

$$\frac{G_{y_t}(t, u^t, p_{y_t}^t) - G_{y_t}(0, u, p_{y_t})}{t} = \frac{G_{y_t}(t, u, p_{y_t}^t) - G_{y_t}(0, u, p_{y_t}^t)}{t}. \quad (62)$$

We want to pass to the limit on the right hand side. To do so notice

$$\frac{\Psi(\Phi_t(y_t), u(y_t)) - \Psi(y_t, u(y_t))}{t} = \int_0^1 \nabla \Psi(s\Phi_t(y_t) + (1-s)y_t, u(y_t)) ds \frac{\Phi_t(y_t) - y_t}{t} \quad (63)$$

and consequently

$$\left| \frac{\Psi(\Phi_t(y_t), u(y_t)) - \Psi(y_t, u(y_t))}{t} - \nabla_y \Psi(y, u(y)) \right| \leq c \underbrace{\left\| \frac{\Phi_t - \text{id}}{t} - X \right\|_{C(\bar{D}, \mathbf{R}^{2,2})}}_{\rightarrow 0, \text{ in view of Lemma 1.10}}. \quad (64)$$

By Lemma 2.14, we obtain  $p_{y_t}^t \rightarrow p_y$  in  $W_{\Gamma, q'}^1(\Omega)$  for  $q' = q/(q-1)$ . Thus Lemma 1.10 implies

$$\begin{aligned} & \int_{\Omega} \frac{\beta(|B(t)\nabla u|^2)A(t) - \beta(|\nabla u|^2)}{t} \nabla p_{y_t}^t \cdot \nabla u + \frac{\xi(t) - 1}{t} p_{y_t}^t u \, dx - \int_{\Omega} \frac{f^t - f}{t} p_{y_t}^t \, dx \\ & \rightarrow \int_{\Omega} \beta(|\nabla u|^2) A'(0) \nabla p_y \cdot \nabla u + 2\beta'(|\nabla u|^2) B'(0) \nabla u \cdot \nabla u \nabla p_y \cdot \nabla u \, dx + \int_{\Omega} \text{div}(X) p_y u - f' p_y \, dx \end{aligned} \quad (65)$$

as  $t \searrow 0$ . Now (64) and (65) together imply (61) and thus our claim.  $\square$

*Proof of Theorem 2.15.* According to (51) we have  $j(\Omega_t^{\Gamma, y}) = G_y(t, u, p_y^t)$  for all  $t$  and all  $y \in \bar{\Omega}$ . So an application of Lemma 2.16 with  $y(t) \equiv y$ , yields  $dj(\Omega^{\Gamma, y})(X) = \frac{d}{dt} G_y(t, u^t, p_y)|_{t=0} = \partial_t G_y(0, u, p_y)$ .  $\square$

Now we can present explicit formulas for the shape derivative of  $j(\cdot)$ .

**Corollary 2.17.** *Let  $\Omega^{\Gamma} \in \Xi$  and assume  $2 < q < q_0$ .*

(a) *The shape derivative of  $j(\cdot)$  at  $\Omega^{\Gamma, y}$ ,  $y \in \bar{\Omega}$ , in direction  $X \in \mathring{C}^1(D, \mathbf{R}^2)$  is given by*

$$dj(\Omega^{\Gamma, y})(X) = \int_{\Omega} \mathbf{S}_1(u, p_y) : \partial X + \mathbf{S}_0(u, p_y) \cdot X \, dx + X(y) \cdot \nabla_y \Psi(y, u(y)), \quad (66)$$

where

$$\begin{aligned} \mathbf{S}_1(u, p_y) &:= (\beta(|\nabla u|^2) \nabla u \cdot \nabla p_y + u p_y - f p_y) I - \beta(|\nabla u|^2) (\nabla u \otimes \nabla p_y + \nabla p_y \otimes \nabla u) \\ &\quad - 2\beta'(|\nabla u|^2) (\nabla u \cdot \nabla p_y) \nabla u \otimes \nabla u \end{aligned} \quad (67)$$

$$\mathbf{S}_0(u, p_y) := -\nabla f p_y. \quad (68)$$

Here,  $(u, p_y) \in W_{\Gamma, q}^1(\Omega) \times W_{\Gamma, q'}^1(\Omega)$  solve (22) and (56), respectively.

(b) *Assume  $\partial\Omega \in C^1$ ,  $u \in H^2(\Omega)$ ,  $f \in H^2(\Omega)$  and  $p_y \in H^2(\Omega \setminus \{y\})$  for  $y \in \Omega$  and  $p_y \in H^2(\Omega)$  for  $y \in \partial\Omega$ . Then for every  $y \in \Omega$ ,*

$$-\text{div}(\mathbf{S}_1(u, p_y)) + \mathbf{S}_0(u, p_y) = 0 \quad \text{a.e. in } \Omega \setminus \{y\} \quad (69)$$

and for every  $y \in \partial\Omega$ ,

$$-\text{div}(\mathbf{S}_1(u, p_y)) + \mathbf{S}_0(u, p_y) = 0 \quad \text{a.e. in } \Omega. \quad (70)$$

Moreover, for all  $y \in \Omega$ ,

$$dj(\Omega^{\Gamma, y})(X) = \int_{\partial\Omega} \mathbf{S}_1(u, p_y) \nu \cdot \nu (X \cdot \nu) \, ds + (\mathbf{S}_1(u, p_y) \nu \otimes \delta_y) X + X(y) \cdot \nabla_y \Psi(y, u(y)), \quad (71)$$

where  $(\mathbf{S}_1(u, p_y)\nu \otimes \delta_y)X := \lim_{\delta \searrow 0} \int_{\partial B_\delta(y)} \mathbf{S}_1(u, p_y)\nu \cdot X \, ds$  and for all  $y \in \partial\Omega$ ,

$$dj(\Omega^{\Gamma, y})(X) = \int_{\partial\Omega} \mathbf{S}_1(u, p_y)\nu \cdot \nu(X \cdot \nu) \, ds + X(y) \cdot \nabla_y \Psi(y, u(y)). \quad (72)$$

Here  $B_\delta(y)$  denotes the ball centered at  $y$  with radius  $\delta$ .

*Proof.* At first by Theorem 2.15,  $j(\Omega^{\Gamma, y})(X) = \partial_t G_y^X(0, u, p_y)$  and

$$\begin{aligned} \partial_t G_y^X(0, u, p_y) &= \int_{\Omega} \beta(|\nabla u|^2) A'(0) \nabla p_y \cdot \nabla u + 2\beta'(|\nabla u|^2) B'(0) \nabla u \cdot \nabla u \nabla p_y \cdot \nabla u \, dx \\ &\quad + \int_{\Omega} \operatorname{div}(X) p_y u - f' p_y \, dx - \int_{\Omega} f' p_y \, dx + X(y) \cdot \nabla_y \Psi(y, u(y)), \end{aligned} \quad (73)$$

for all  $X \in \dot{C}^1(D, \mathbf{R}^2)$ , where according to Lemma 1.10,  $A'(0) = \operatorname{div}(X)I - \partial X - \partial X^\top$  and  $f' := \operatorname{div}(X)f + \nabla f \cdot X$ . Therefore it is readily verified that (73) can be brought into the tensor form (66).

Let us now prove that (66) is in fact equivalent to (71) when  $u \in H^2(\Omega)$ ,  $f \in H^2(\Omega)$  and  $p_y \in H^2(\Omega)$  for  $y \in \partial\Omega$  and  $p_y \in H^2(\Omega \setminus \{y\})$  for  $y \in \Omega$ . Let  $y \in \Omega$  be given and choose  $\delta > 0$  such that  $\overline{B_\delta(y)} \subset \Omega$  and define the Lipschitz domain  $\Omega_\delta := \Omega \setminus \overline{B_\delta(y)}$ . By Nagumo's theorem it follows that for all  $\delta > 0$  so that  $\overline{B_\delta(y)} \subset \Omega$ ,  $dj(\Omega^{\Gamma, y})(X) = 0$  for all  $X \in C_c^1(\Omega_\delta, \mathbf{R}^2)$ . But according to (66) this is equivalent to

$$\int_{\Omega} \mathbf{S}_1(u, p_y) : \partial X + \mathbf{S}_0(u, p_y) \cdot X \, dx = 0 \quad \forall X \in C_c^1(\Omega_\delta, \mathbf{R}^2). \quad (74)$$

Now partial integration and the fundamental theorem of the calculus of variations yield for all small  $\delta > 0$   $-\operatorname{div}(\mathbf{S}_1(u, p_y)) + \mathbf{S}_0(u, p_y) = 0$  a.e. on  $\Omega_\delta$  and hence

$$-\operatorname{div}(\mathbf{S}_1(u, p_y)) + \mathbf{S}_0(u, p_y) = 0 \quad \text{a.e. on } \Omega \setminus \{y\}. \quad (75)$$

Setting  $\mathbf{S}_1 := \mathbf{S}_1(u, p_y)$  and  $\mathbf{S}_0 := \mathbf{S}_0(u, p_y)$ , we obtain

$$\begin{aligned} dj(\Omega^{\Gamma, y})(X) &= \int_{B_\delta(y)} \mathbf{S}_1 : \partial X + \mathbf{S}_0 \cdot X \, dx + \int_{\Omega_\delta} \underbrace{(-\operatorname{div}(\mathbf{S}_1) + \mathbf{S}_0)}_{=0, (75)} \cdot X \, dx + \int_{\partial\Omega} \mathbf{S}_1 \nu \cdot X \, ds \\ &\quad + X(y) \cdot \nabla_y \Psi(y, u(y)) + \int_{\partial B_\delta(y)} \mathbf{S}_1 \nu \cdot X \, ds \quad \text{for } X \in C_c^1(D, \mathbf{R}^2). \end{aligned} \quad (76)$$

Further Hölder's inequality shows

$$\left| \int_{B_\delta(y)} \mathbf{S}_1 : \partial X + \mathbf{S}_0 \cdot X \, dx \right| \leq |B_\delta(y)|^{1/q'} \|X\|_{C^1} (\|\mathbf{S}_1\|_{L_q(\Omega, \mathbf{R}^{2,2})} + \|\mathbf{S}_0\|_{L_q(\Omega, \mathbf{R}^2)})$$

and the right hand side goes to zero as  $\delta \searrow 0$ . Consequently

$$\lim_{\delta \searrow 0} \int_{\partial B_\delta(y)} \mathbf{S}_1 \nu \cdot X \, ds = dj(\Omega^{\Gamma, y})(X) - \int_{\partial\Omega} \mathbf{S}_1 \nu \cdot X \, ds + X(y) \cdot \nabla_y \Psi(y, u(y)) \quad (77)$$

for all  $X \in \dot{C}^1(D, \mathbf{R}^2)$ . Observe  $X \mapsto (\mathbf{S}_1(u, p_y)\nu \otimes \delta_y)X := \lim_{\delta \searrow 0} \int_{\partial B_\delta(y)} \mathbf{S}_1 \nu \cdot X \, ds$  is linear and continuous as a mapping  $\dot{C}^1(D, \mathbf{R}^2) \rightarrow \mathbf{R}$ . Define  $X_\tau := X - (X \cdot \tilde{\nu})\tilde{\nu}$ , where  $\tilde{\nu}$  is a smooth extension of  $\nu$  such that  $\operatorname{supp} \tilde{\nu} \subset D \setminus \overline{B_\delta(y)}$  and  $X \in C_c^1(D, \mathbf{R}^2)$ . Then  $dj(\Omega^{\Gamma, y})(X_\tau) = 0$  which is equivalent to

$$\int_{\partial\Omega} \mathbf{S}_1 \nu \cdot X \, ds = \int_{\partial\Omega} \mathbf{S}_1 \nu \cdot \nu(X \cdot \nu) \, ds \quad \text{for all } X \in C_c^1(D, \mathbf{R}^2). \quad (78)$$

So inserting (78) into (76), we recover (71).

Now let  $y \in \partial\Omega$ . Then  $dj(\Omega^{\Gamma, y})(X) = 0$  for all  $X \in C_c^1(\Omega, \mathbf{R}^2)$  and this is equivalent to  $\int_{\Omega} \mathbf{S}_1(u, p_y) : \partial X + \mathbf{S}_0(u, p_y) \cdot X \, dx = 0$  for all  $X \in C_c^1(\Omega, \mathbf{R}^2)$ , from which we conclude by partial integration and

the fundamental theorem of calculus of variations,  $-\operatorname{div}(\mathbf{S}_1(u, p_y)) + \mathbf{S}_0(u, p_y) = 0$  a.e. on  $\Omega$ . Hence integrating by parts we obtain

$$dj(\Omega^{\Gamma, y})(X) = \int_{\partial\Omega} \mathbf{S}_1 \nu \cdot X \, ds + X(y) \cdot \nabla_y \Psi(y, u(y)) \quad \text{for } X \in \mathring{C}^1(D, \mathbf{R}^2) \quad (79)$$

and since also in this case (78) is valid we get (72).  $\square$

**Remark 2.18.** Notice that if we strengthen the assumption in item (b) of the previous theorem and assume for all  $y \in \Omega$ ,  $p_y \in H^2(\Omega)$ , then it follows from (77) by partial integration  $(\mathbf{S}_1(u, p_y) \nu \otimes \delta_y)X = 0$ .

## 2.5 Eulerian semi-derivative of $J_\infty(\cdot)$

The following theorem is a Danksin type theorem and follows essentially from the proof of [7, Theorem 2.1, p.524]. Since our setting is different from the one in the book we give a proof.

**Lemma 2.19.** Let  $K \subset \mathbf{R}^d$  be a compact set,  $\tau > 0$  a positive number and  $g : [0, \tau] \times K \rightarrow \mathbf{R}$  some function. Define for  $t \in [0, \tau]$  the set  $R^t = \{z \in K : \max_{x \in K} g(t, x) = g(t, z)\}$  with the convention  $R := R^0$ . Assume that

(A1) for all  $x \in R$ , the partial derivative  $\partial_t g(0^+, x)$  exists,

(A2) for all  $t \in [0, \tau]$ , the function  $x \mapsto g(t, x)$  is upper semi-continuous,

(A3) for all real nullsequences  $(t_n)$ ,  $t_n \searrow 0$ , and all sequences  $(y_{t_n})$ ,  $y_{t_n} \in R^{t_n}$  converging to some  $y \in R$ , we have

$$\lim_{n \rightarrow \infty} \frac{g(t_n, y_{t_n}) - g(0, y_{t_n})}{t_n} = \partial_t g(0^+, y). \quad (80)$$

Then

$$\frac{d}{dt} \left( \max_{x \in K} g(t, x) \right)_{t=0} = \max_{x \in R} \partial_t g(0^+, x). \quad (81)$$

*Proof.* Due to assumption (A2) and the compactness of  $K$  the set  $R^t$  is nonempty for all  $t \in [0, \tau]$ . Furthermore, by definition, for all  $t \geq 0$ ,  $y_t \in R^t$ , and  $y \in R$  we have  $g(t, y_t) \geq g(t, y)$  and  $g(0, y) \geq g(0, y_t)$ . Using these two inequalities we obtain  $g(t, y_t) - g(0, y) \geq g(t, y) - g(0, y)$  and also  $g(t, y_t) - g(0, y) \leq g(t, y_t) - g(0, y_t)$  and consequently

$$g(t, y) - g(0, y) \leq g(t, y_t) - g(0, y) \leq g(t, y_t) - g(0, y_t). \quad (82)$$

Setting  $\delta(t) := (g(t, y_t) - g(0, y))/t$  it is sufficient to show that  $\liminf_{t \searrow 0} \delta(t) = \limsup_{t \searrow 0} \delta(t)$  and one of the limits is finite. By assumption (A1) and (82), we obtain the chain of inequalities  $\partial_t g(0^+, y) \leq \liminf_{t \searrow 0} \delta(t) \leq \limsup_{t \searrow 0} \delta(t)$  for all  $y \in R$ . Since the previous inequality is true for all  $y \in R$  it implies

$$\max_{y \in R} \partial_t g(0^+, y) \leq \liminf_{t \searrow 0} \delta(t) \leq \limsup_{t \searrow 0} \delta(t). \quad (83)$$

Now  $K$  is compact and  $y_t \in K$  for all  $t \geq 0$ , so we find for each nullsequence  $(t_n)$  a subsequence, still indexed the same, and  $y \in K$ , such that  $y_{t_n} \rightarrow y$  as  $n \rightarrow \infty$ . We need to show that  $y \in R$ . In fact it follows for all  $x \in K$ ,  $g(t_n, x) \leq g(t_n, y_{t_n}) = g(0, y_{t_n}) + t_n \frac{g(t_n, y_{t_n}) - g(0, y_{t_n})}{t_n}$  and thus using Assumption (A2) we get for all  $x \in K$ ,

$$g(0, x) = \limsup_{n \rightarrow \infty} g(t_n, x) \leq \limsup_{n \rightarrow \infty} g(0, y_{t_n}) + \limsup_{n \rightarrow \infty} t_n \frac{g(t_n, y_{t_n}) - g(0, y_{t_n})}{t_n} \leq g(0, y). \quad (84)$$

This shows that  $y$  is a maximum of  $g(0, \cdot)$ , that is,  $y \in R$ . We deduce from (82) and Assumption (A3),  $\limsup_{t \searrow 0} \delta(t) \leq \lim_{n \rightarrow \infty} \frac{g(t_n, y_{t_n}) - g(0, y_{t_n})}{t_n} = \partial_t g(0^+, y)$  and hence  $\limsup_{t \searrow 0} \delta(t) \leq \partial_t g(0^+, y) \leq \max_{y \in R} \partial_t g(0^+, y)$ . Finally combining the previous inequality with (83) yields,  $\max_{y \in R} \partial_t g(0^+, y) \leq \liminf_{t \searrow 0} \delta(t) \leq \limsup_{t \searrow 0} \delta(t) \leq \max_{y \in R} \partial_t g(0^+, y)$  and thus the desired result.  $\square$

Let us define the set

$$R(\Omega^\Gamma) := \{x \in \bar{\Omega} : J_\infty(\Omega^\Gamma) = \Psi(x, u(x))\}. \quad (85)$$

We can now prove the following main result.

**Theorem 2.20.** *Let  $\Omega^\Gamma \in \Xi$  be given and suppose  $q \in (2, q_0]$ . Then the Eulerian semi-derivative of the shape function  $J_\infty$  given by (12) at  $\Omega^\Gamma$  in direction  $X \in \dot{C}^1(D, \mathbf{R}^2)$  is given by*

$$dJ_\infty(\Omega^\Gamma)(X) = \max_{y \in R(\Omega^\Gamma)} \partial_t G_y^X(0, u, p_y), \quad (86)$$

where  $(u, p_y) \in W_{\Gamma, q}^1(\Omega) \times W_{\Gamma, q'}^1(\Omega)$  solve (22) and (56), respectively.

*Proof.* We apply Lemma 2.19 with  $g(t, y) := G_y(t, u^t, p_y) = G_y(t, u, p_y^t)$  and  $K := \bar{\Omega}$ . Assumption (A1) is clear. Assumptions (A2) follows from Lemma 2.14 and the continuity of  $u$ . Assumption (A3) is a consequence of Lemma 2.16. Thus all assumptions are satisfied and the claim follows.  $\square$

The next theorem gives a complete characterisation of the Eulerian semi-derivative of  $J_\infty(\cdot)$ . We show that the Eulerian semi-derivative is related to the maximum of a boundary integral provided the state and adjoint state are more result. This can be seen as a generalisation of [20, Proposition 3.2].

**Theorem 2.21.** *Suppose  $\Omega^\Gamma \in \Xi$  and  $2 < q < q_0$ .*

(a) *The Eulerian semi-derivative of the maximum function (12) at  $\Omega^\Gamma$  in direction  $X \in \dot{C}^1(D, \mathbf{R}^2)$  is given by*

$$dJ_\infty(\Omega^\Gamma)(X) = \max_{y \in R(\Omega^\Gamma)} \left( \int_{\Omega} \mathbf{S}_1(u, p_y) : \partial X + \mathbf{S}_0(u, p_y) \cdot X \, dx + X(y) \cdot \nabla_y \Psi(y, u(y)) \right), \quad (87)$$

where  $\mathbf{S}_1, \mathbf{S}_0$  are defined in (68), (67). The adjoint state  $p_y \in W_{\Gamma, q'}^1(\Omega)$  solves for  $y \in R(\Omega^\Gamma)$ ,

$$\int_{\Omega} b(x, u) L p_y \cdot L \varphi \, dx = -\partial_u \Psi(y, u(y)) \varphi(y) \quad \text{for all } \varphi \in W_{\Gamma, q}^1(\Omega), \quad (88)$$

and the state  $u \in W_{\Gamma, q}^1(\Omega)$  solves the state equation (22). Moreover,

$$\int_{\Omega} \mathbf{S}_1(u, p_y) : \partial X + \mathbf{S}_0(u, p_y) \cdot X \, dx + X(y) \cdot \nabla_y \Psi(y, u(y)) \leq 0 \quad (89)$$

for all  $X \in \dot{C}^1(\Omega, \mathbf{R}^2)$  and for all  $y \in R(\Omega^\Gamma)$ .

(b) *When  $\partial\Omega \in C^1$ ,  $u \in H^2(\Omega)$ ,  $p_y \in H^2(\Omega \setminus \{y\})$  for all  $y \in \Omega \cap R(\Omega^\Gamma)$  and  $p_y \in H^2(\Omega)$  for all  $y \in \partial\Omega \cap R(\Omega^\Gamma)$ , then (87) is equivalent to*

$$dJ_\infty(\Omega^\Gamma)(X) = \max_{y \in R(\Omega^\Gamma)} \left( \int_{\partial\Omega} \mathbf{S}_1(u, p_y) \nu \cdot \nu X_\nu \, ds + \chi_{\partial\Omega}(y) X_\nu(y) \nu(y) \cdot \nabla_y \Psi(y, u(y)) \right) \quad (90)$$

where  $X \in \dot{C}^1(D, \mathbf{R}^2)$  and  $X_\nu := X \cdot \nu$ . Here  $\chi_{\partial\Omega}$  denotes the characteristic function associated with  $\partial\Omega$ .

*Proof.* Equation (87) follows by combining Corollary 2.17, Theorem 2.20 and Theorem 2.15.

We now prove (89). By Nagumo's theorem it follows  $dJ_\infty(\Omega^\Gamma)(X) = 0$  for all  $X \in \dot{C}^1(\Omega, \mathbf{R}^2)$  and this implies,

$$dj(\Omega^{\Gamma, y})(X) \leq dJ_\infty(\Omega^\Gamma)(X) = 0 \quad (91)$$

for all  $y \in R(\Omega^\Gamma)$  and all  $X \in \dot{C}^1(\Omega, \mathbf{R}^2)$ . Taking into account (66) we recover (89).

Now under the assumption of item (b) we know from Corollary 2.17 that  $dj(\Omega^{\Gamma, y})$  has the form (71) for  $y \in R(\Omega^\Gamma) \cap \Omega$ . So inserting (71) into (91) and taking into account  $X = 0$  on  $\partial\Omega$ , we obtain  $(\mathbf{S}_1(u, p_y) \nu \otimes \delta_y) X + X(y) \cdot \nabla_y \Psi(y, u(y)) \leq 0$  for all  $y \in \Omega \cap R(\Omega^\Gamma)$  and all  $X \in \dot{C}^1(\Omega, \mathbf{R}^2)$ . Since this inequality is true for all  $X \in \dot{C}^1(\Omega, \mathbf{R}^2)$  we obtain  $(\mathbf{S}_1(u, p_y) \nu \otimes \delta_y) + \nabla_y \Psi(y, u(y)) = 0$  for all  $y \in \Omega \cap R(\Omega^\Gamma)$ . Further we get  $X(y) \cdot \nabla_y \Psi(y, u(y)) \leq 0$  for all  $y \in \partial\Omega \cap R(\Omega^\Gamma)$  and all  $X \in C^1(\Omega, \mathbf{R}^2)$  with  $X \cdot \nu = 0$  on  $\partial\Omega$ . Consequently (90) follows from (71) and (72).  $\square$

**Corollary 2.22.** *Let  $\Omega^\Gamma \in \Xi$ . Assume that  $R(\Omega^\Gamma) \subset \partial\Omega$  and  $\Gamma = \emptyset$ . Then*

$$dJ_\infty(\Omega^\Gamma)(X) = \max_{y \in R(\Omega^\Gamma)} X(y) \cdot \nabla_y \Psi(y, u(y)) \quad \text{for all } X \in \dot{C}^1(D, \mathbf{R}^2). \quad (92)$$

*Proof.* This follows immediately from (87), since  $p_y = 0$  for all  $y \in \Gamma_0 = \partial\Omega$ .  $\square$

**Corollary 2.23.** *Let  $\Omega^\Gamma \in \Xi$ . If  $R(\Omega^\Gamma) = \{y_0\}$  is a single-tone, then  $J_\infty(\cdot)$  is shape differentiable at  $\Omega^\Gamma$ .*

**Corollary 2.24.** *Let  $\Omega^\Gamma \in \Xi$ . We have  $dJ_\infty(\Omega^\Gamma)(X + Y) \leq dJ_\infty(\Omega^\Gamma)(X) + dJ_\infty(\Omega^\Gamma)(Y)$  for all  $X, Y \in \dot{C}^1(D, \mathbf{R}^2)$ .*

**Remark 2.25.** *We note that to show the differentiability of  $J_\infty(\cdot)$  one might want to use the material derivative approach; cf. [24]. In our general setting this approach is difficult to apply as one would have to show the strong differentiability of  $t \mapsto u^t$  from  $[0, \tau]$  into  $W_{\Gamma, p}^1(\Omega)$  for some  $q > 2$ . The weak differentiability is not sufficient as  $W_{\Gamma, p}^1(\Omega)$  does not embed compactly into  $C_\Gamma(\Omega)$ .*

### 3 Characterisation of stationary points of $J_\infty(\cdot)$

This section is devoted to the characterisation of stationary points of the shape function  $J_\infty(\cdot)$ . We closely follow the approach of [8], where finite dimensional problems are studied. Accordingly many results have to be carefully modified to account for the infinite dimensionality of our problem. Throughout this section we suppose that the assumptions of Theorem 2.20, item (a), are satisfied.

#### 3.1 Gradient of $j(\cdot)$

Let  $\mathcal{H}(\Omega, \mathbf{R}^2)$  be some Hilbert space of functions from  $\Omega$  into  $\mathbf{R}^2$ . According to Corollary 2.17 the shape derivative of  $j(\cdot)$  at  $\Omega^{\Gamma, y}$ ,  $y \in \bar{\Omega}$  is given by

$$dj(\Omega^{\Gamma, y})(X) = \int_{\Omega} \mathbf{S}_1(u, p_y) : \partial X + \mathbf{S}_0(u, p_y) \cdot X \, dx + X(y) \cdot \nabla_y \Psi(y, u(y)) \quad (93)$$

for all  $X \in \dot{C}^1(D, \mathbf{R}^2)$ . Assume that  $\mathcal{H}(\Omega, \mathbf{R}^2) \subset \dot{C}^{0,1}(D, \mathbf{R}^2)$ , such that  $a \otimes \delta_y : \mathcal{H}(\Omega, \mathbf{R}^2) \rightarrow \mathbf{R} : X \mapsto a \cdot X(y)$  is continuous for all  $a \in \mathbf{R}^2$ . Denoting by  $\mathcal{R} : \mathcal{H}(\Omega, \mathbf{R}^2) \rightarrow (\mathcal{H}(\Omega, \mathbf{R}^2))^*$  the Riesz isomorphism then we have  $\mathcal{R}^{-1}(dj(\Omega^{\Gamma, y})) = \nabla j(\Omega^{\Gamma, y})$  and by definition it satisfies for fixed  $y \in \bar{\Omega}$  the variational equation,

$$(\nabla j(\Omega^{\Gamma, y}), \varphi)_{\mathcal{H}} = dj(\Omega^{\Gamma, y})(\varphi) \quad \text{for all } \varphi \in \mathcal{H}(\Omega, \mathbf{R}^2). \quad (94)$$

As a consequence (87) can be written as  $dJ_\infty(\Omega^\Gamma)(X) = \max_{y \in R(\Omega^\Gamma)} (\nabla j(\Omega^{\Gamma, y}), X)_{\mathcal{H}}$ . One way to construct the space  $\mathcal{H}(\Omega, \mathbf{R}^2)$  is to define it as reproducing kernel Hilbert space associated with matrix-valued kernels of the form  $K(x, y) = \phi(|x - y|^2/\sigma)I$ ,  $\sigma > 0$ , where  $\phi \in C^1(\mathbf{R})$  is some smooth function. Then [9, Lemma 3.13] provides an explicit formula for the gradient of  $j(\Omega^{\Gamma, y})$ . An alternative way, also described in [9], is to choose  $\mathcal{H}(\Omega, \mathbf{R}^2)$  as a finite element space  $V_N(\Omega, \mathbf{R}^2)$ . This is described in more detail in the last section of this paper. In the following we fix the space  $\mathcal{H}(\Omega, \mathbf{R}^2)$  and denote the gradient of  $j(\cdot)$  simply by  $\nabla j(\Omega^{\Gamma, y})$  always keeping in mind that it depends on the choice of the space  $\mathcal{H}(\Omega, \mathbf{R}^2)$  and the inner product chosen.

#### 3.2 Stationary points

The following presentation is based on [8, Chapter 3]. We point out that there only the finite dimensional case was studied and we have to adapt our results to the infinite dimensional setting.

Let us begin with the definition stationary points.

**Definition 3.1.** *The set  $\Omega^\Gamma \in \Xi$  is said to be a stationary point for  $J_\infty(\cdot)$  with respect to perturbations in  $\mathcal{H}(\Omega, \mathbf{R}^2)$  if  $dJ_\infty(\Omega^\Gamma)(X) \geq 0$  for all  $X \in \mathcal{H}(\Omega, \mathbf{R}^2)$ .*



Define the sets

$$H(\Omega^\Gamma) := \{\nabla j(\Omega^\Gamma, y) : y \in R(\Omega^\Gamma)\} \quad (95)$$

and the convex hull of  $H(\Omega^\Gamma)$  by

$$L(\Omega^\Gamma) := \left\{ \sum_{k=1}^n \alpha_k X^k : n \in \mathbf{N}, X^k \in H(\Omega^\Gamma), \alpha_k \geq 0, k = 1, \dots, n, \sum_{k=1}^n \alpha_k = 1 \right\}. \quad (96)$$

The closure of  $L(\Omega^\Gamma)$  in  $\mathcal{H}(\Omega, \mathbf{R}^2)$  is denoted by  $\bar{L}(\Omega^\Gamma)$ . We now show that  $H(\Omega^\Gamma)$  is closed and bounded.

**Remark 3.2.** Notice that the set  $\bar{L}(\Omega^\Gamma)$  is related to the Clarke subdifferential; cf. [4].

**Lemma 3.3.** The set  $H(\Omega^\Gamma)$  is closed and bounded in  $\mathcal{H}(\Omega, \mathbf{R}^2)$ .

*Proof.* We first show that  $H(\Omega^\Gamma)$  is closed. Set  $X^y := \nabla j(\Omega^\Gamma, y)$ . Let  $\{y_n\}$  be a sequence in  $R(\Omega^\Gamma)$  such that  $X^{y_n} \rightarrow X$  in  $\mathcal{H}(\Omega, \mathbf{R}^2)$ . Since  $R(\Omega^\Gamma)$  is compact there is a subsequence  $\{y_{n_k}\}$  such that  $y_{n_k} \rightarrow y \in R(\Omega^\Gamma)$  as  $k \rightarrow \infty$ . Hence Lemma 2.14 implies  $p_{n_k} \rightharpoonup p_y$  weakly in  $W_{\Gamma, q'}^1(\Omega)$  and using (94), we get

$$\begin{aligned} (X^{y_{n_k}}, \varphi)_{\mathcal{H}} &= \left( \int_{\Omega} \mathbf{S}_1(u, p_{y_{n_k}}) : \partial \varphi + \mathbf{S}_0(u, p_{y_{n_k}}) \cdot \varphi \, dx + \nabla_y \Psi(y_{n_k}, u(y_{n_k})) \cdot \varphi(y_{n_k}) \right) \\ &\rightarrow \left( \int_{\Omega} \mathbf{S}_1(y, p_y) : \partial \varphi + \mathbf{S}_0(u, p_y) \cdot \varphi \, dx + \nabla_y \Psi(y, u(y)) \right) \\ &= (X, \varphi)_{\mathcal{H}} \quad \text{for all } \varphi \in \mathcal{H}(\Omega, \mathbf{R}^2). \end{aligned} \quad (97)$$

Now as the weak limit and the strong limit coincide it follows  $X = X^y$ . The boundedness of  $H(\Omega^\Gamma)$  is obvious since  $\bar{\Omega} \rightarrow \mathbf{R} : y \mapsto \|X^y\|_{\mathcal{H}}$  is continuous and  $\bar{\Omega}$  compact.  $\square$

With the definition of  $H(\Omega^\Gamma)$  we can write the Eulerian semi-derivative of  $J_\infty(\cdot)$  as  $dJ_\infty(\Omega^\Gamma)(X) = \max_{Z \in H(\Omega^\Gamma)} (Z, X)_{\mathcal{H}}$  and the right hand side can be further rewritten.

**Lemma 3.4.** There holds for all  $X \in \mathcal{H}(\Omega, \mathbf{R}^2)$ ,

$$\max_{Z \in H(\Omega^\Gamma)} (Z, X)_{\mathcal{H}} = \max_{Z \in \bar{L}(\Omega^\Gamma)} (Z, X)_{\mathcal{H}}. \quad (98)$$

*Proof.* Since  $H(\Omega^\Gamma) \subset \bar{L}(\Omega^\Gamma)$ , we immediately get the inequality

$$\max_{Z \in H(\Omega^\Gamma)} (Z, X)_{\mathcal{H}} \leq \max_{Z \in \bar{L}(\Omega^\Gamma)} (Z, X)_{\mathcal{H}}. \quad (99)$$

To show the other inequality let  $\hat{Z} \in \bar{L}(\Omega^\Gamma)$ . Then by definition we find a sequence  $\{Z_n\}$  in  $\mathcal{H}(\Omega, \mathbf{R}^2)$  such that  $Z_n \rightarrow \hat{Z}$  in  $\mathcal{H}(\Omega, \mathbf{R}^2)$  and  $Z_n = \sum_{i=1}^n \alpha_n^i Z^{y_i^n}$  with  $\sum_{i=1}^n \alpha_n^i = 1$ ,  $\alpha_n^i \geq 0$ . We obtain for all  $n \geq 1$ ,

$$(Z_n, \varphi)_{\mathcal{H}} = \sum_{i=1}^n \alpha_n^i (Z^{y_i^n}, \varphi)_{\mathcal{H}} \leq \max_{Z \in H(\Omega^\Gamma)} (Z, \varphi)_{\mathcal{H}} \sum_{i=1}^n \alpha_n^i = \max_{Z \in H(\Omega^\Gamma)} (Z, \varphi)_{\mathcal{H}}. \quad (100)$$

Passing to the limit  $n \rightarrow \infty$  shows  $(\hat{Z}, \varphi)_{\mathcal{H}} \leq \max_{Z \in H(\Omega^\Gamma)} (Z, \varphi)_{\mathcal{H}}$  for all  $\hat{Z} \in \bar{L}(\Omega^\Gamma)$ . Taking the supremum over  $\hat{Z}$  and taking into account inequality (99) finishes the proof.  $\square$

**Lemma 3.5.** The set  $\Omega^\Gamma \in \Xi$  is a stationary point for  $J_\infty(\cdot)$  in  $\mathcal{H}(D, \mathbf{R}^2)$ , that is,  $dJ_\infty(\Omega^\Gamma)(X) \geq 0$  for all  $X \in \mathcal{H}(\Omega, \mathbf{R}^2)$  if and only if  $0 \in \bar{L}(\Omega^\Gamma)$ .

*Proof.* According to Lemma 1.12, we have  $0 \notin \bar{L}(\Omega^\Gamma)$  if and only if there exists  $X_0 \in \bar{L}(\Omega^\Gamma)$ ,  $X_0 \neq 0$  satisfying  $(X_0, X_0)_{\mathcal{H}(\Omega, \mathbf{R}^d)} \geq (X_0, \varphi)_{\mathcal{H}(\Omega, \mathbf{R}^d)}$  for all  $\varphi \in \bar{L}(\Omega^\Gamma)$ . Thus  $0 \notin \bar{L}(\Omega^\Gamma)$  implies  $-\|X_0\|_{\mathcal{H}(\Omega, \mathbf{R}^d)}^2 \geq \max_{\varphi \in \bar{L}(\Omega^\Gamma)} (-X_0, \varphi)_{\mathcal{H}(\Omega, \mathbf{R}^d)} = dJ_\infty(\Omega^\Gamma)(-X_0)$  and hence  $dJ_\infty(\Omega^\Gamma)(-X_0) < 0$ . Conversely if there exists  $X_0 \in \bar{L}(\Omega^\Gamma)$ ,  $X_0 \neq 0$ , such that  $dJ_\infty(\Omega^\Gamma)(X_0) < 0$ , then  $(X_0, \varphi)_{\mathcal{H}(\Omega, \mathbf{R}^d)} \leq dJ_\infty(\Omega^\Gamma)(X_0) < 0$  for all  $\varphi \in \bar{L}(\Omega^\Gamma)$  which can only be true if  $0 \notin \bar{L}(\Omega^\Gamma)$ . This finishes the proof.  $\square$

**Definition 3.6.** We call  $\mathbf{g} \in \mathcal{H}(\Omega, \mathbf{R}^2)$  with  $\|\mathbf{g}\|_{\mathcal{H}} = 1$  steepest descent direction of  $J_{\infty}(\cdot)$  at  $\Omega$  if

$$dJ_{\infty}(\Omega^{\Gamma})(\mathbf{g}) \leq dJ_{\infty}(\Omega^{\Gamma})(\varphi) \quad \text{for all } \varphi \in \mathcal{H}(\Omega, \mathbf{R}^2) \text{ with } \|\varphi\|_{\mathcal{H}} = 1. \quad (101)$$

**Remark 3.7.** Since according [17, Lemma 2.8] the Eulerian semi-derivative  $dJ_{\infty}(\Omega^{\Gamma})(\cdot)$  is 1-homogeneous we can restrict ourselves to the unique sphere in the previous definition.

At this juncture let us introduce for  $\Omega^{\Gamma} \in \Xi$  the function

$$\psi(\Omega^{\Gamma}) := \min_{\substack{X \in \mathcal{H}(\Omega, \mathbf{R}^2) \\ \|X\|_{\mathcal{H}} = 1}} \max_{Z \in H(\Omega^{\Gamma})} (Z, X)_{\mathcal{H}}. \quad (102)$$

**Lemma 3.8.** Suppose that  $\psi(\Omega^{\Gamma}) < 0$ . Then

$$\psi(\Omega^{\Gamma}) = \max_{Z \in \bar{L}(\Omega^{\Gamma})} \left( Z, -\frac{\hat{Z}}{\|\hat{Z}\|_{\mathcal{H}}} \right)_{\mathcal{H}} = -\|\hat{Z}\|_{\mathcal{H}}, \quad (103)$$

where  $\hat{Z} \in \mathcal{H}$  solves the minimisation problem  $\|\hat{Z}\|_{\mathcal{H}} = \min_{X \in \bar{L}(\Omega^{\Gamma})} \|X\|_{\mathcal{H}}$ .

*Proof.* We apply Lemma 1.12 with  $H := \mathcal{H}(\Omega, \mathbf{R}^2)$ ,  $x_0 = 0$  and  $K = \bar{L}(\Omega^{\Gamma})$ . Hence we find  $Z^* \neq 0$  in  $\bar{L}(\Omega^{\Gamma})$  with  $(Z, Z^*)_{\mathcal{H}} \geq (Z^*, Z^*)_{\mathcal{H}}$  for all  $Z \in \bar{L}(\Omega^{\Gamma})$ . Therefore for  $\bar{G} := Z^*/\|Z^*\|_{\mathcal{H}}$  it holds  $(\bar{G}, Z)_{\mathcal{H}} \leq -\|Z^*\|_{\mathcal{H}}$  for all  $Z \in \bar{L}(\Omega^{\Gamma})$  which in turn implies

$$\max_{Z \in \bar{L}(\Omega^{\Gamma})} (Z^*, Z)_{\mathcal{H}} = \left( Z^*, -\frac{Z^*}{\|Z^*\|_{\mathcal{H}}} \right)_{\mathcal{H}} = -\|Z^*\|_{\mathcal{H}}. \quad (104)$$

This is already the second equality in (103). As for the second one we observe that Cauchy-Schwarz's inequality shows  $(Z, Y)_{\mathcal{H}} \geq -\|Z\|_{\mathcal{H}}\|Y\|_{\mathcal{H}}$  for all  $Z, Y \in \mathcal{H}(\Omega, \mathbf{R}^2)$ . Hence we get  $-\|Z^*\|_{\mathcal{H}} \leq (Z^*, G)_{\mathcal{H}} \leq \max_{Z \in \bar{L}(\Omega^{\Gamma})} (Z^*, G)_{\mathcal{H}}$  for arbitrary  $G \in \mathcal{H}(\Omega, \mathbf{R}^2)$  with  $\|G\|_{\mathcal{H}} = 1$ . So combining the previous inequality with (104) yields the desired result.  $\square$

**Lemma 3.9.** The function  $l(G) := \max_{Z \in \bar{L}(\Omega^{\Gamma})} (Z, G)_{\mathcal{H}}$  attains its minimum on the unit sphere in exactly one point.

*Proof.* The proof of [8, Hilfsatz 3.3.7. p.63-44] applies also to our setting.  $\square$

In analogy to the case in which Eulerian semi-derivative is linear (see [9, Lemma 2.10]) we can prove the existence and uniqueness of steepest descent directions in the nonlinear case. However, the computation is more involved than in the linear case.

**Theorem 3.10.** Let  $\Omega^{\Gamma} \in \Xi$  and suppose  $\psi(\Omega^{\Gamma}) < 0$ . Then there is a unique steepest descent direction  $\mathbf{g} \in \mathcal{H}(\Omega, \mathbf{R}^2)$ ,  $\|\mathbf{g}\|_{\mathcal{H}} = 1$ , for  $J_{\infty}(\cdot)$  at  $\Omega^{\Gamma}$  given by  $\mathbf{g} := -\frac{\hat{Z}}{\|\hat{Z}\|_{\mathcal{H}}}$ , where  $\hat{Z} = P_{\bar{L}(\Omega^{\Gamma})}(0)$  is the projection of 0 (in  $\mathcal{H}(\Omega, \mathbf{R}^2)$ ) onto  $\bar{L}(\Omega^{\Gamma})$ .

*Proof.* Follows from Lemmas 3.9 and 3.8.  $\square$

### 3.3 $\epsilon$ -stationary points

In order to define stable numerical algorithms we introduce  $\epsilon$ -stationary points. For  $\epsilon \geq 0$  we define

$$R_{\epsilon}(\Omega^{\Gamma}) := \{y \in \bar{\Omega} : J_{\infty}(\Omega^{\Gamma}) - \Psi(y, u(\Omega, \Gamma, y)) \leq \epsilon\} \quad (105)$$

$$H_{\epsilon}(\Omega^{\Gamma}) := \{\nabla j(\Omega^{\Gamma, y}) : y \in R_{\epsilon}(\Omega^{\Gamma})\} \quad (106)$$

and the convex hull of  $H_{\epsilon}(\Omega^{\Gamma})$  is denoted by  $L_{\epsilon}(\Omega^{\Gamma})$ . Let us introduce for  $\epsilon \geq 0$ ,

$$\psi_{\epsilon}(\Omega^{\Gamma}) := \min_{\substack{X \in \mathcal{H}(\Omega, \mathbf{R}^2) \\ \|X\|_{\mathcal{H}} = 1}} \max_{y \in R_{\epsilon}(\Omega^{\Gamma})} (\nabla j(\Omega^{\Gamma, y}), X)_{\mathcal{H}}.$$

Analogously to steepest descent directions we introduce  $\epsilon$ -steepest descent directions.

**Definition 3.11.** (i) We call  $\Omega^\Gamma \in \Xi$  an  $\epsilon$ -stationary point for  $J_\infty(\cdot)$  with respect to perturbations in  $\mathcal{H}(\Omega, \mathbf{R}^2)$  if  $\psi_\epsilon(\Omega^\Gamma) \geq 0$ .

(ii) We call  $\mathbf{g}_\epsilon$  a  $\epsilon$ -steepest descent direction for  $J_\infty(\cdot)$  at  $\Omega^\Gamma$  in  $\mathcal{H}(\Omega, \mathbf{R}^2)$  if

$$\max_{Z \in \bar{L}_\epsilon(\Omega^\Gamma)} (Z, \mathbf{g}_\epsilon)_\mathcal{H} = \min_{\substack{X \in \mathcal{H}(\Omega, \mathbf{R}^2) \\ \|X\|_\mathcal{H}=1}} \max_{Z \in \bar{L}_\epsilon(\Omega^\Gamma)} (Z, X)_\mathcal{H}. \quad (107)$$

It is readily verified that  $\Omega^\Gamma$  is an  $\epsilon$ -stationary point if and only if  $0 \in \bar{L}_\epsilon(\Omega^\Gamma)$ . Moreover, if  $\psi_\epsilon(\Omega^\Gamma) < 0$  there is a unique  $\epsilon$ -steepest descent direction at  $\Omega^\Gamma$  in the space  $\mathcal{H}(\Omega, \mathbf{R}^2)$  given by  $\mathbf{g} = -\frac{\hat{\mathbf{g}}_\epsilon}{\|\hat{\mathbf{g}}_\epsilon\|_\mathcal{H}}$ , where  $\mathbf{g}_\epsilon$  is the projection of 0 onto  $\bar{L}_\epsilon(\Omega^\Gamma)$ .

The crucial point of  $\epsilon$ -steepest descent directions  $\mathbf{g}_k$  is that they decrease the cost function  $J_\infty(\cdot)$ . Suppose that  $\psi_{\epsilon_k}(\Omega^\Gamma) < 0$  and let  $\mathbf{g}_k := \mathbf{g}_k^\epsilon$  be the  $\epsilon_k$ -steepest descent direction. Then

$$dJ_\infty(\Omega^\Gamma)(\mathbf{g}_k) \leq \max_{Z \in \bar{L}_{\epsilon_k}(\Omega^\Gamma)} (Z, \mathbf{g}_k)_\mathcal{H} < 0 \quad (108)$$

and as a consequence  $J_\infty(\Phi_t^{\mathbf{g}_k}(\Omega^\Gamma)) < J_\infty(\Omega^\Gamma)$  for sufficiently small  $t$ . The parameter  $\epsilon$  is a sort of regularisation parameter and ensures that the steepest descent directions are not “too local”.

### 3.4 Discrete problems

#### Discretisation of the domain $\Omega$

We assume that  $\Omega$  is a polygonal set. Let  $\{\mathcal{T}_h\}_{h>0}$  denote a family of simplicial triangulations  $\mathcal{T}_h = \{K\}$  consisting of triangles  $K$  such that

$$\bar{\Omega} = \bigcup_{K \in \mathcal{T}_h} K, \quad \forall h > 0.$$

For every element  $K \in \mathcal{T}_h$ ,  $h(K)$  stands for the diameter of  $K$  and  $\rho(K)$  for the diameter of the largest ball contained in  $K$ . The maximal diameter of all elements is denoted by  $h$ , i.e.,  $h := \max\{h(K) \mid K \in \mathcal{T}_h\}$ . Each  $K \in \mathcal{T}_h$  consists of three nodes and three edges and we denote the set of nodes and edges by  $\mathcal{N}_h$  and  $\mathcal{E}_h$ , respectively. We assume that there exists a positive constant  $\varrho > 0$ , independent of  $h$ , such that  $\frac{h(K)}{\rho(K)} \leq \varrho$  holds for all elements  $K \in \mathcal{T}_h$  and all  $h > 0$ .

#### Discrete $\epsilon$ -steepest descent directions and the quadratic program

In order to obtain an algorithm we select for fixed  $\epsilon > 0$  a finite subset  $R_\epsilon^h(\Omega^\Gamma) \subset R_\epsilon(\Omega^\Gamma)$  of points. We use the triangulation of  $\Omega$  as discretisation, that is,

$$R_\epsilon^h(\Omega^\Gamma) := R_\epsilon(\Omega^\Gamma) \cap \mathcal{N}_h = \{y_1, \dots, y_{N_\epsilon^h}\}. \quad (109)$$

We have  $\#(R_\epsilon^h(\Omega)) = N_\epsilon^h$ . Let us set  $H_\epsilon^h(\Omega) := \{\nabla j(\Omega^\Gamma, y) : y \in R_\epsilon^h(\Omega)\}$  and denote by  $L_\epsilon^h(\Omega)$  the convex hull of  $H_\epsilon^h(\Omega)$ . For  $y \in R_\epsilon^h(\Omega)$  we introduce the vectors  $X_k := \nabla j(\Omega^\Gamma, y_k)$  and order them  $\{X_1, \dots, X_{N_\epsilon^h}\}$ . For simplicity set henceforth  $N := N_\epsilon^h$  and keep in mind that  $N$  depends on  $\epsilon$  and  $h$ . In order to obtain steepest descent directions in  $\mathcal{H}(\Omega, \mathbf{R}^2)$ , we need to solve  $\min_{X \in L_\epsilon^h(\Omega)} \|X\|_\mathcal{H}$ . Using the definition of  $L_\epsilon^h(\Omega)$  we see that this task is equivalent to solving the quadratic problem

$$\min \sum_{k,l=1}^N \alpha_k \alpha_l (X_k, X_l)_\mathcal{H} \quad \text{subject to} \quad \sum_{k=1}^N \alpha_k = 1, \quad \alpha_k \geq 0. \quad (110)$$

Defining  $Q_N := ((X_k, X_l)_\mathcal{H})_{l,k=1,\dots,N}$ ,  $G_N := (1, \dots, 1)$ ,  $E_N := -I$ ,  $g_N = -(1, \dots, 1)^\top$  and  $\alpha = (\alpha_1, \dots, \alpha_N)^\top$ , problem (110) can be written in the canonical form:  $\min_\alpha Q_N \alpha \cdot \alpha$  subject to  $B_N \alpha = 0$  and  $E_N \alpha \leq g_N$ . This quadratic problem is convex and thus admits a unique solution  $\alpha^* = (\alpha_1^*, \dots, \alpha_N^*)$ . The  $\epsilon$ -steepest descent direction is given by

$$\mathbf{g}_\epsilon^h := -\frac{\hat{\mathbf{g}}_\epsilon^*}{\|\hat{\mathbf{g}}_\epsilon^*\|_\mathcal{H}}, \quad \text{where} \quad \hat{\mathbf{g}}_\epsilon^* := \sum_{k=1}^N \alpha_k^* \nabla j(\Omega^\Gamma, y_k). \quad (111)$$

**Remark 3.12.** *It is clear that we are not obliged to use the triangulation of  $\Omega$  to construct a discretisation for  $R_\epsilon^h(\Omega^\Gamma)$ , however, it is advantageous from the practical point of view.*

## 4 Numerical realisation

### 4.1 Problem setting

We consider two cost functions

$$J_\infty(\Omega) := \max_{x \in \bar{\Omega}} |u(x) - u_d(x)|^2, \quad J_2(\Omega) := \int_{\Omega} |u - u_d|^2 dx, \quad (112)$$

where in either case  $u$  is the solution of  $(x = (x_1, x_2))$

$$\begin{aligned} -\Delta u(x) + u(x) &= (2\pi^2 + 1) \sin(\pi x_1) \sin(\pi x_2) \quad \text{in } \Omega, \\ u(x) &= 0 \quad \text{on } \partial\Omega. \end{aligned} \quad (113)$$

Notice that we set  $\Omega := \Omega^\emptyset$  since  $\Gamma = \emptyset$ . We now define  $u_d(x) := \sin(\pi x_1) \sin(\pi x_2)$ , such that by construction  $\Omega_{opt} \in \text{argmin} J_2$  and  $\Omega_{opt} \in \text{argmin} J_\infty$  with  $\Omega_{opt} := (0, 1) \times (0, 1)$ . Indeed the unique solution of (113) on  $(0, 1) \times (0, 1)$  reads  $u(x) = \sin(\pi x_1) \sin(\pi x_2)$  as can be readily verified. By the properties of the sinus function we see that also every other domain  $\Omega_n := (2n, 2n+1) \times (2n, 2n+1)$ ,  $n \in \mathbf{Z}$  is a global minimum of  $J_\infty$  and  $J_2$ , respectively.

### Finite element approximation

Now we describe discretisations of  $dJ_2(\Omega)$  and  $dJ_\infty(\Omega)$ . Let  $V_h(\Omega)$ ,  $h > 0$ , denote the usual  $H^1(\Omega)$  conforming finite element space, that is,

$$V_h(\Omega) := \{v \in C(\bar{\Omega}) : v|_K \in \mathcal{P}^1(K), \forall K \in \mathcal{T}_h\}. \quad (114)$$

By  $\mathring{V}_h(\Omega)$  we denote all function of the space  $V_h(\Omega)$  that vanish on the boundary  $\partial\Omega$ . For each  $y \in \bar{\Omega}$  the finite element approximation  $(u_h, p_h^y) \in \mathring{V}_h(\Omega) \times \mathring{V}_h(\Omega)$  of state (22) and adjoint state equation (56) reads,

$$\int_{\Omega} \nabla u_h \cdot \nabla \varphi + u_h \varphi dx = \int_{\Omega} f \varphi dx \quad \text{for all } \varphi \in \mathring{V}_h(\Omega) \quad (115)$$

$$\int_{\Omega} \nabla \varphi \cdot \nabla p_{y,h} + \varphi p_{y,h} dx = -2(u(y) - u_h(y)) \varphi(y) \quad \text{for all } \varphi \in \mathring{V}_h(\Omega). \quad (116)$$

With the discretised state and adjoint state equation the discretised version of (87), (where  $\Psi(y, \zeta) := |\zeta - u_d(y)|^2$ ) reads

$$dJ_\infty^h(\Omega)(X) = \max_{y \in R^h(\Omega)} \left( \int_{\Omega} \mathbf{S}_1^{y,h} : \partial X + \mathbf{S}_0^{y,h} \cdot X dx - X(y) \cdot \nabla u_d(y) 2(u_h(y) - u_d(y)) \right), \quad (117)$$

where for  $y \in R_\epsilon^h(\Omega)$  we set  $\mathbf{S}_1^{y,h} := \mathbf{S}_1(u_h, p_{y,h})$  and  $\mathbf{S}_0^{y,h} := \mathbf{S}_0(u_h, p_{y,h})$  with  $\mathbf{S}_1, \mathbf{S}_2$  being defined in (67),(68) (with  $\beta \equiv 1$ ).

The shape derivative of  $J_2(\Omega) = \int_{\Omega} |u - u_d|^2 dx$ , subject to  $u$  solves (113), in an open and bounded subset  $\Omega \subset D$  in direction  $X \in \mathring{C}^1(D, \mathbf{R}^2)$  (see [26] for the computation) is given by

$$dJ_2(\Omega)(X) = \int_{\Omega} \mathbf{T}_1(u, \hat{p}) : \partial X + \mathbf{T}_0(u, \hat{p}) \cdot X dx, \quad (118)$$

where  $\hat{p}$  solves the adjoint equation

$$\int_{\Omega} \nabla \hat{p} \cdot \nabla \varphi + \hat{p} \varphi dx = - \int_{\Omega} 2(u - u_d) \varphi dx \quad \text{for all } \varphi \in \mathring{H}^1(\Omega). \quad (119)$$

The tensors  $\mathbf{T}_1$  and  $\mathbf{T}_2$  are given by  $\mathbf{T}_1(u, \hat{p}) := (|u - u_d|^2 - f\hat{p} + \nabla u \cdot \nabla \hat{p})I - (\nabla u \otimes \nabla \hat{p} + \nabla \hat{p} \otimes \nabla u)$  and  $\mathbf{T}_0(u, \hat{p}) := -\nabla f\hat{p} - 2\nabla u_d(u - u_d)$ . The discrete version of (118) reads

$$dJ_2^h(\Omega)(X) = \int_{\Omega} \mathbf{T}_1(u_h, \hat{p}_h) : \partial X + \mathbf{T}_0(u_h, \hat{p}_h) \cdot X \, dx, \quad (120)$$

where the discrete state  $u_h$  solves (115) and the discrete adjoint state  $\hat{p}_h \in \mathring{V}_h(\Omega)$  solves:

$$\int_{\Omega} \nabla \hat{p}_h \cdot \nabla \varphi + \hat{p}_h \varphi \, dx = - \int_{\Omega} 2(u_h - u_d) \varphi \, dx \quad \text{for all } \varphi \in \mathring{V}_h(\Omega). \quad (121)$$

#### 4.1.1 Choice of the metric

We run our numerical tests with two different metrics on the space  $V_h(\Omega) \times V_h(\Omega)$ , namely the  $H^1$  metric and the Euclidean metric. Let  $v^1, \dots, v^{2N}$  be a basis of  $V_h(\Omega) \times V_h(\Omega)$  and  $\alpha_i, \beta_i$  in  $\mathbf{R}$ ,  $i, j = 1, 2, \dots, 2N$  and suppose  $v = \sum_{i=1}^{2N} \alpha_i v^i$  and  $w = \sum_{i=1}^{2N} \beta_i v^i$ . The  $H^1$  metric and Euclidean metric are defined by

$$(v, w)_{H^1} := \sum_{i,j=1}^{2N} \alpha_i \beta_j M_{ij}, \quad (v, w)_{V_h} := \sum_{i,j=1}^{2N} \alpha_i \beta_j \delta_{ij},$$

where  $M_{ij}$  is defined by  $M_{ij} = \int_{\Omega} \partial v^i : \partial v^j + v^i \cdot v^j \, dx$  and  $\delta_{ij}$  denotes the Kronecker delta. We denote by  $\mathcal{H}_{\text{Sob}}$  and  $\mathcal{H}_{\text{Euc}}$  the space  $V_h(\Omega) \times V_h(\Omega)$  equipped with the  $H^1$  and Euclidean metric, respectively. Both spaces are kernel reproducing Hilbert spaces and thus the point evaluation is continuous; see [9, Section 3].

The approximated Eulerian semi-derivative (117) can equivalently be written as:

$$dJ_{\infty}^h(\Omega)(X) = \max_{y \in R^h(\Omega)} (\nabla^{\text{Euc}} j^h(\Omega^y), X)_{\mathcal{H}_{\text{Euc}}} = \max_{y \in R^h(\Omega)} (\nabla^{\text{Sob}} j^h(\Omega^y), X)_{\mathcal{H}_{\text{Sob}}}, \quad (122)$$

where for all  $y \in R^h(\Omega)$  the gradient  $\nabla^{\text{Sob}} j^h(\Omega^y)$  is defined as the solution of

$$(\nabla^{\text{Sob}} j^h(\Omega^y), \varphi)_{\mathcal{H}_{\text{Sob}}} = \int_{\Omega} \mathbf{S}_1^{y,h} : \partial \varphi + \mathbf{S}_0^{y,h} \cdot \varphi \, dx - 2\nabla u_d(y) \cdot \varphi(y)(u_h(y) - u_d(y)) \quad (123)$$

for all  $\varphi \in V_h(\Omega) \times V_h(\Omega)$ . The gradient  $\nabla^{\text{Euc}} j^h(\Omega^y)$  is explicitly given by

$$\nabla^{\text{Euc}} j^h(\Omega^y) = \sum_{k=1}^{2N} \left( \int_{\Omega} \mathbf{S}_1^{y,h} : \partial v^k + \mathbf{S}_0^{y,h} \cdot v^k \, dx - 2\nabla u_d(y) \cdot v^k(y)(u_h(y) - u_d(y)) \right) v^k. \quad (124)$$

We refer to [9, Section 3] for more details. The advantage of the Euclidean metric is that it does not require the solution of a variational problem but only the evaluation of the shape derivative  $dj^h(\Omega^y)(v^j)$  at the basis elements  $v^j$ .

Similarly, the discretised shape derivative of  $J_2$  can be written as

$$dJ_2^h(\Omega)(X) = (\nabla^{\text{Euc}} J_2^h(\Omega), X)_{\mathcal{H}_{\text{Euc}}} = (\nabla^{\text{Sob}} J_2^h(\Omega^y), X)_{\mathcal{H}_{\text{Sob}}}, \quad (125)$$

where

$$(\nabla^{\text{Sob}} J_2^h(\Omega), \varphi)_{\mathcal{H}_{\text{Sob}}} = \int_{\Omega} \mathbf{T}_1^h : \partial \varphi + \mathbf{T}_0^h \cdot \varphi \, dx \quad \text{for all } \varphi \in V_h(\Omega) \times V_h(\Omega) \quad (126)$$

and

$$\nabla^{\text{Euc}} J_2^h(\Omega) = \sum_{k=1}^{2N} \left( \int_{\Omega} \mathbf{T}_1^h : \partial v^k + \mathbf{T}_0^h \cdot v^k \, dx \right) v^k. \quad (127)$$

We will use the notation  $J_2^h(\Omega) := \int_{\Omega} |u_h - u_d|^2 \, dx$  and  $J_{\infty}^h(\Omega) = \max_{x \in \bar{\Omega}} |u_h(x) - u_d(x)|^2$ .

## 4.2 Steepest descent algorithm

The following algorithm uses the discretisation described in Section 3.4.

**Data:** Let  $n = 0$ ,  $h > 0, \gamma > 0$  and  $N \in \mathbf{N}$  be given. Initialise domain  $\Omega_0 \subset \mathbf{R}^2$ . Let  $N_2 > 0$ .

**while**  $n \leq N$  **do**

- 1.) choose  $\epsilon$ , so that,  $\#(R^h(\Omega_n)) \leq \#(R_\epsilon^h(\Omega_n)) \leq N_2 + \#(R^h(\Omega_n))$  ;
- 2.) solve (115) to get  $u_h$  and (116) for all  $y \in R_\epsilon^h(\Omega_n)$  to obtain  $p_{y,h}$ ;
- 3.) solve (123) to obtain gradients  $\{\nabla j(\Omega^{y_1}), \dots, \nabla j(\Omega^{y_{N_\epsilon^h}})\}$ ;
- 4.) solve quadratic program (110) to obtain  $\mathbf{g}_\epsilon^h$  defined in (111);
- 5.) decrease  $t$  until

$$J_\infty^h((\text{id} + t\mathbf{g}_\epsilon^h)(\Omega_n)) < J_\infty^h(\Omega_n) \quad (128)$$

and set  $\Omega_{n+1} := (\text{id} + t\mathbf{g}_\epsilon^h)(\Omega_n)$  ;

**if**  $J_\infty^h(\Omega_n) - J_\infty^h(\Omega_{n+1}) \geq \gamma(J_\infty^h(\Omega_0) - J_\infty^h(\Omega_1))$ ;

**then**

- | increase  $n \rightarrow n + 1$  and continue program;

**else**

- | abort algorithm, no sufficient decrease

**end**

**end**

**Algorithm 1:**  $\epsilon$ -steepest descent algorithm

## 4.3 Numerical simulations

The state equation, adjoint state equation and the shape derivative are discretised as described in (115), (116) and (117), respectively. The domain  $\Omega$  consists in each iteration of around 5500 nodes and we remesh in each iteration step. The boundary  $\partial\Omega$  itself is discretised with a fixed number of 400 nodes which are moved during the optimisation process. As initial domain we choose a circle centered at  $x = (0.5, 0.5)$  with radius  $r = \sqrt{6} \approx 2.44$ .

In Figure 1 and Figure 2 several iterations of Algorithm 1 applied to  $J_\infty^h(\cdot)$  are displayed. The blue points indicate points of the triangulation contained in  $R_\epsilon^h(\Omega_n)$ , where  $n$  is the current iteration number. The number  $N_2$  in Algorithm 1 is chosen to be between 40 and 80. We did not perform a linesearch, that means, step four in the algorithm is replace by choosing a constant step size. It can be seen that the optimal shape is quite good approximated using: (i) the  $H^1$  metric in Figure 1 and (ii) the Euclidean metric in Figure 2. Even the corners are reconstructed quite well. Observe that the points in  $R_\epsilon^h(\Omega_n)$  are mostly distributed on the boundary  $\partial\Omega$ , so that for those points no adjoint equation has to be computed (cf. Corollary 2.22). In Figure 3 and Figure 4 we applied [9, Algorithm 1] to the cost function  $J_2^h$ . We use the same discretisation as before.

In Figure 6 the values  $J_2^h(\Omega_n)$  over the number of iteration are plotted both in log scale. For the dashed lines we used the  $H^1$  metric and for the solid lines the Euclidean metric. It makes sense to replace  $J_\infty(\Omega_n)$  by  $J_2^h(\Omega_n)$  as a measure for the convergence rate as the latter cost function can be estimated by  $J_\infty(\Omega_n)$  using Hölder's inequality. We observe that the convergence rate in the smooth case (minimising  $J_2^h(\cdot)$ )  $J_2^h(\Omega_n)$  is slower than in the nonsmooth case (minimising  $J_\infty^h(\cdot)$ )  $J_2^h(\Omega_n)$ . In the nonsmooth case the convergence rate even speeds up again in later iterations. That in the nonsmooth case corners do not perfectly match might have the reason that the  $\epsilon$  in our algorithm does not tend to zero in the end as we keep  $N_2 \geq 40$ . In fact it is difficult to find a reasonable condition to decrease  $\epsilon$ . In the numerical practice it seems better to keep the number  $N_2$  fixed in order to obtain a stable algorithm. The  $H^1$  metric yields smoother shapes than the Euclidean metric in general.

Notice that to compute one descent direction for  $J_\infty^h$  we have to solve at least one state equation,  $\#(R_\epsilon^h(\Omega_n)) - \#(\Gamma_0)$  adjoint state equations and  $\#(R_\epsilon^h(\Omega_n))$  gradient equations  $\nabla j(\Omega_n^y)$ . However the computation is *perfectly parallel*, that means, the computation of the adjoints and gradients can be parallized. Another possibility to reduce the computational cost is to use the boundary expression (90), but the accuracy of this expression is lower than the domain expression (87). In fact after discretisation (90) and (87) are not equivalent anymore; cf. [9]. In contrast, to compute a steepest descent direction for  $J_2^h$  only one state equation, one adjoint equation and one shape gradient has to be computed.

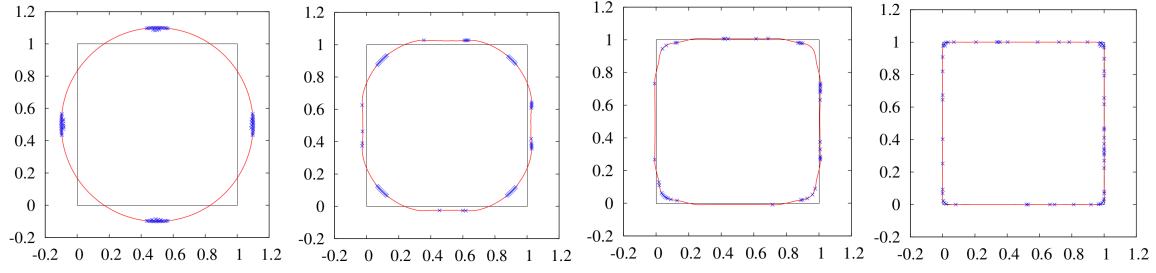


Figure 1: Results for  $J_\infty^h$  with  $H^1$  metric; blue: points in  $R_\epsilon^h(\Omega_n)$ ; red: boundary of shape  $\partial\Omega_n$ ; from left to right: initial shape, iteration 10, 120, 2000

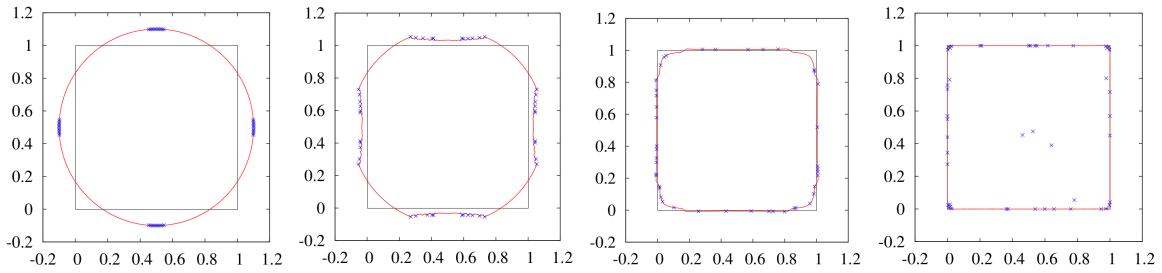


Figure 2: Results for  $J_\infty^h$  with Euclidean metric; blue: points in  $R_\epsilon^h(\Omega_n)$ ; red: boundary of shape  $\partial\Omega_n$ ; from left to right: initial shape, iteration 10, 100, 2000

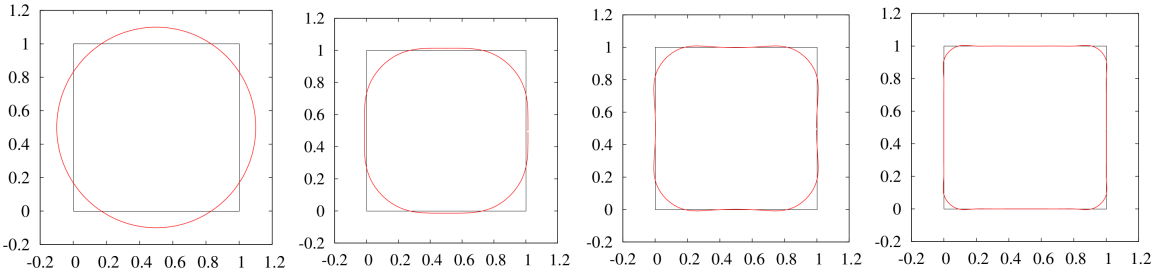


Figure 3: Results for  $J_2^h$  with  $H^1$  metric; from left to right: initial shape, iteration 20, 120, 2000

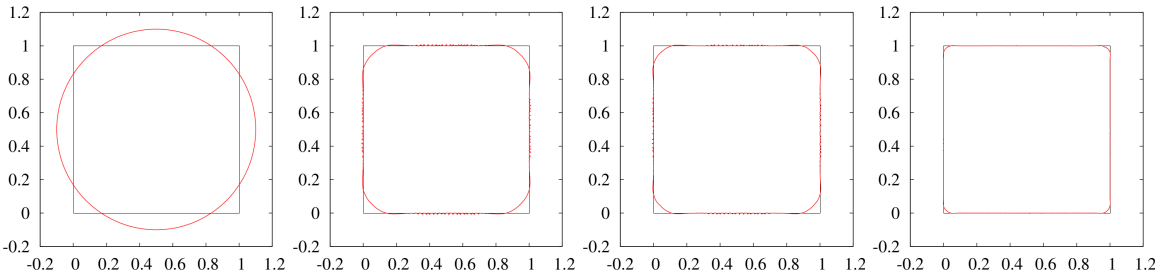


Figure 4: Results for  $J_2^h$  with Euclidean metric; from left to right: initial shape, iteration 60, 120, 2000

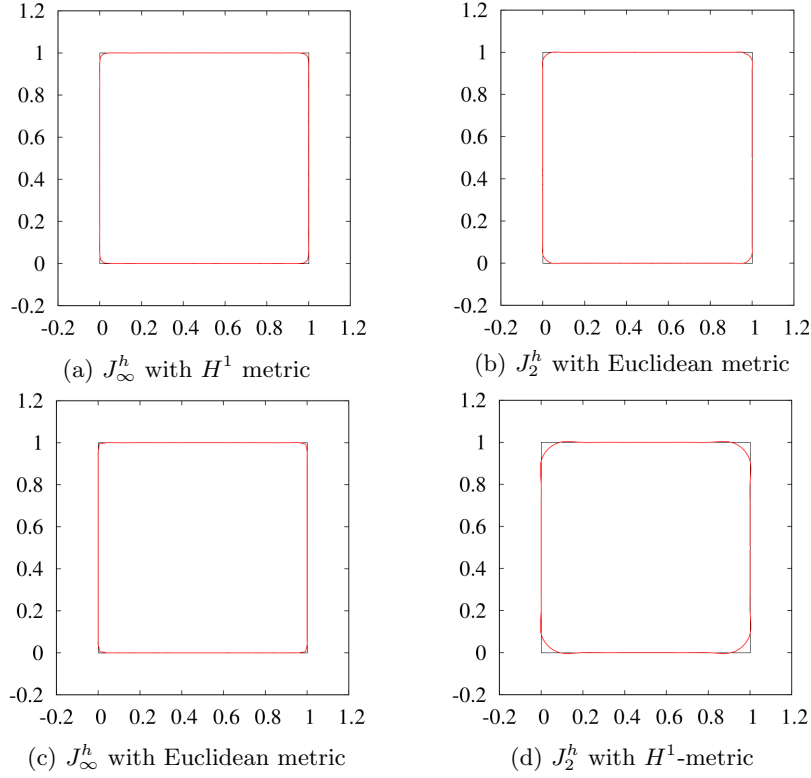


Figure 5: Comparison of final shapes

All implementations were carried out within the FEniCS Software package [10]. The quadratic program (110) is solved with the python package *cvxopt*; cf. [1].

Figure 6: x-axis number of iterations and y-axis values  $J_2^h(\Omega_n)$

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